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The One and the Dyad: the Foundations of Ancient Mathematics[1]

What Exists Instead of Infinite Space in Euclid's *Elements*?

ABSTRACT: This paper contains a new interpretation of Euclidean geometry. It is argued that ancient Euclidean geometry was created in a quite different intuitive model (or frame), without infinite space, infinite lines and surfaces. This ancient intuitive model of Euclidean geometry is reconstructed in connection with Plato's unwritten doctrine. The model creates a kind of "hermeneutical horizon" determining the explicit content and mathematical methods used. In the first section of the paper, it is argued that there are no actually infinite concepts in Euclid's *Elements*. In the second section, it is argued that ancient mathematics is based on Plato's highest principles: the One and the Dyad and the role of *agrapha dogmata* is unveiled.
KEYWORDS: Euclidean geometry • Euclid's *Elements* • ancient mathematics • Plato's unwritten doctrine • philosophical hermeneutics • history of science and mathematics • philosophy of mathematics • philosophy of science

The possibility to imagine all of the theorems from Euclid's *Elements* in a quite different intuitive framework is interesting from the philosophical point of view.

I can give one example showing how it was possible to lose the genuine image of Euclidean geometry. In many translations into Latin (Heiberg) and English (Heath) of the theorems from Book X, the term "area" (*spatium*) is used. For instance, the translations of X. 26 are as follows: „Spatium medium non excedit medium spatio rationali" and "A medial area does not exceed a medial area by a rational area"[2]. However, in the corresponding Greek

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[2] Cf. J.L. Heiberg, *Euclidis Elementa*, Teubner, Leipzig 1883–1888. Retrieved 2007, from: http://www.perseus.tufts.edu/cgi-bin/ptext?lookup=Euc.+toc or from http://www.wilbourhall.

text, i.e. "μέσον μέσου ούχ ὑπερέχει ῥητῷ", there is no word for "area". The problem does not lie in the introduction of such an auxiliary term, but with the change of exact meaning which this word has for us and had for Euclid. For us, this word means "every possible area". For Euclid it meant only "a rectilinear figure which is equal to a medial area", and "a medial area" has a meaning determined not only by the definition from X. 26 but also by definitions X. 1 – X. 4 (cf. especially Def. X. 4[3]), i.e. in the connection with the basic line ("assigned line"). "Medial" is a local property only. Therefore, Euclid's theorems are not as general as they seem to be.

On the other hand, the meaning of the word χωρίον (i.e. a particular place, room, area, a space enclosed by lines, a rectangle, *etc.*) used in many theorems is preserved correctly in the translations; cf. for instance X. 55 or X. 46. This is because, in every case, this word is used in the context of the classified straight lines. Thus, such theorems as X. 26 contain a brachylogy which makes it possible to interpret the text in two different ways.

org/; J.L. Heiberg, *Euclidis opera omnia*, J.L. Heiberg & H. Menge (Eds.), Bibliotheca Scriptorum Graecorum et Romanorum Teubneriana, vols. I–VIII, in Aedibus B.G. Teubneri, Lipsiae 1883–9, (vol. I: *Euclidis Elementa, Libros I–IV Continens*, edidit et latinae interpretatus est J.L. Heiberg, Lipsiae 1883; vol. II: *Euclidis Elementa, Libros V–IX Continens*, Edidit Et Latinae Interpretatus Est J.L. Heiberg, Lipsiae 1884; vol. III: *Euclidis Elementa, Librum X Continens*, Edidit Et Latinae Interpretatus Est J.L. Heiberg, Lipsiae 1886; vol. IV: *Euclidis Elementa, Libros XI–XIII Continens*, edidit et latinae interpretatus est J.L. Heiberg, Lipsiae 1885; vol. V: *Continens Elementorum Qui Peruntur Libros XIV–XV Et Scholia In Elementa Cum Prolegmenis Criticis Et Appendicibus*, edidit et latinae interpretatus est J.L. Heiberg, Lipsiae 1888; vol. VI: *Euclidis Data Cum Commentario Marini Et Scholiis Antiquis*, Edidit Henrikus Menge, Lipsiae 1896; vol. VII: *Euclidis Optica, Opticorum Recensio Theonis, Catoptrica Cum Scholiis Antiquis*, Edidit J.L. Heiberg, Lipsiae 1895; Vol. VIII: *Suplementum: Anaritii In Decem Libros Elementorum Euclidis Commentarii*, Edidit Maximilianus Curtze, Lipsiae 1899, (retrieved 2009, from http://www.wilbourhall.org/)). Cf. also T.L. Heath, *The thirteen books of Euclid's `Elements' translated from the text of Heiberg with introduction and commentary*, vols. 1–3, Cambridge 1908; retrieved 2008, from http://www.wilbourhall.org/pdfs/, vol. 3, Book X.

3 The Greek text of Def. X. 4 is as follows: "δ′. Καὶ τὸ μὲν ἀπὸ τῆς προτεθείσης εὐθείας τετράγωνον ῥητόν, καὶ τὰ τούτῳ σύμμετρα ῥητά, τὰ δὲ τούτῳ ἀσύμμετρα ἄλογα καλείσθω, καὶ ἁι δυνάμεναι αὐτὰ ἄλογοι, εἰ μὲν τετράγωνα εἴη, αὐταὶ αἱ πλευραί, εἰ δὲ ἕτερά τινα εὐθύγραμμα, αἱ ἴσα αὐτοῖς τετράγωνα ἀναγράφουσαι". An English translation is (the words in parentheses are added by the translator): "4. And let the square on the assigned straight-line be called rational. And (let areas) commensurable with it (also be called) rational. But (let areas) incommensurable with it (be called) irrational, and (let) their square roots (also be called) irrational — the sides themselves, if the (areas) are squares, and the (straight-lines) describing squares equal to them, if the (areas) are some other rectilinear (figure)"; cf. Euclid, *Euclid's Elements of Geometry.*, R. Fitzpatrick (Ed.), Richard Fitzpatrick (2007). Retrieved 2008, from http:// farside.ph.utexas.edu/euclid.html, transl. by R. Fitzpatrick.

For the Greek mathematicians, to speak about every possible area equal to the given figure was a problem, because "every area" is a "Platonic" concept in the modern sense; i.e. it designates a non-constructive totality of objects. The Greeks were able to speak only about areas equal to the given area if there was a general method of construction of such an area for every given figure of the given kind of figures. For instance, we know very well of the problem of the squaring of the circle. In connection with theorem I. 45 ("To construct a parallelogram equal to a given rectilinear figure in a given rectilinear angle"), Proclus adds:

> It is my opinion that this problem is what led the ancients to attempt the squaring of the circle. For if parallelogram can be found equal to any rectilinear figure, it is worth inquiring whether it is not possible to prove that a rectilinear figure is equal to a circular area. Indeed Archimedes proved that a circle is equal to a right-angled triangle when its radius is equal to one of the sides about the right angle and its perimeter is equal to the base[4].

However, in the times of Euclid, the problem was still open.

For us, it is obvious without any proof that, for every given square, there must exist a circle equal to this square, but ancient geometry was not "continuous" and metrical. We will see that the lack of an "axiom of continuity" is by no means an imprecision or oversight[5]. It is the next fundamental difference between ancient and modern "Euclidean" geometry. In ancient constructive geometry, the allowed methods of constructions, i.e. the postulates, are the most important. The axioms are secondary: it was a well-known fact in Antiquity that there are many properties used in Euclid's proofs which are not listed as axioms. Ancient Euclidean geometry is not a deductive axiomatic system, but a constructive project.

A similar change of meaning is observed regarding the use of the term "line": for us, it means mainly "an infinite straight line", for Euclid, it meant only a "finite straight line". The extension of the meanings of Euclid's text seems to be allowable from the viewpoint of modern geometry, because we still have true theorems. However, our modern theorems are different from

4 Proclus, *A Commentary On the First Book of Euclid's Elements*, G.R. Morrow (ed.), Princeton, NJ 1992, (in Greek: *idem*, *Procli Diadochi in Primum Elementorum Librum Commentarii*, G. Friedlein (ed.), Leipzig 1873, repr. G. Olms, Hildesheim 1967; retrieved 2009, from http://www.wilbourhall. org/). pp. 335, 422,24–423,6; cf. also Archimedes' Proposition I in *Measurement of a Circle*.

5 Cf. also Z. Król, *Platon i podstawy matematyki współczesnej. Pojęcie liczby u Platona*, (*Plato and the Foundations of Modern Mathematics. The Concept of Number by Plato*), Nowa Wieś 2005.

Euclid's theorems, and the proofs from Books X and XIII do not prove what we expect them to prove.

E. Grant writes:

> There is nothing in Euclid's geometry to suggest that he assumed an independent, infinite, three-dimensional, homogeneous space in which the figures of his geometry were located. In a purely geometric sense, such a space would be superfluous because every geometric figure has its own internal space[6].

However, Grant's presentation avoids the problem of the introduction of infinite space into geometry and mathematics. Hermann Weyl also (quoted by Grant) understood that ancient geometry was not about absolute infinite space.

Therefore, to explain the necessity of the use of a new model of understanding of Euclid's *Elements*, I have to explain what we have instead of infinite space, infinite surfaces and infinite straight lines.

1. The absence of infinite geometrical concepts: an infinite line, surface and space

It seems that the concept of an infinite straight line is present in *Elements*, because many Euclidean proofs have counterparts in the infinite intuitive geometrical model, and can be translated with the use of infinite concepts. However, not every Euclidean proof which uses the general and unspecified concept of a "line", is a valid proof for such an infinite counterpart.

Postulate I. 2[7] is concerned with the operation of the prolongation of the given finite straight line. By this Postulate, one can extend every line in one (*ἐπὶ τὰ αὐτὰ μέρη*) or two directions (*ἐφ'ἑκάτερα τὰ μέρη*). Every line, not only a straight line, has boundaries which are points[8]. A line is called infinite (*ἄπειρον*) if it has indefinite and not explicitly indicated boundaries, i.e. it is finite but not exactly defined. There is no place in *Elements* in which an actually infinite straight line, infinite surface or space are used. Everywhere, only finite lines which can be extended in one or in both directions are used.

For example, the extension of lines in one direction is used in the theorems or proofs of I. 5, 14, 16, 17, 20, 21, 27, 29, 31, 32, and — in two direc-

[6] E. Grant, *Much Ado About Nothing. Theories of Space and Vacuum from the Middle Ages to the Scientific Revolution*, Cambridge/London et al. 1982, p. 16.

[7] "To produce a finite straight line continuously in a straight line", Καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ'ἐὐθείας ἐκβαλεῖν .

[8] Cf. Def. I. 3.

tions — in I. 37 and I. 38. In every case, the construction is completed when the finite lines are drawn or produced. The extensions used in constructions are always finite and terminate at a specific point; cf. "Let *BC* be extended to *H*"[9].

It is impossible to construct an actually infinite line. Indefinite lines can only be r e p r e s e n t e d by the specific finite lines used in geometrical proofs and constructions.

However, in every modern translation, this fact is blurred or lost. It can be observed, for instance, in case of theorem I. 12, i.e. "To draw a straight line perpendicular to a given infinite straight line from a given point which is not on it". Our modern understanding of the text is automatically connected with an actually infinite straight line. Proclus, in his *Commentary*, explains in connection with I. 12 that:

> If the line were not infinite, it would be possible so to take the point that it would lie outside the given line, but on a straight line with it, so that the line when prolonged would fall on it; and thus the problem could not be solved. For this reason he posits the straight line as infinite, so that if the point is taken only on one or the other side of the line, there will be no place left in which it can lie in a straight line with the given straight line and thus will lie outside and not on it. This is the reason, then, why the line to which the perpendicular is to be dropped is given as infinite. But it is worth inquiring in what sense in general the infinite line has existence. It is clear that, if there is an infinite line, there will also be an infinite plane, and infinite in actuality if the problem is to be a real one. […] But neither is it possible that there should be an infinite of this sort among separate and indivisible ideas […] But the understanding from which our ideas and demonstrations proceed does not use the infinite for the purpose of knowing it, for the infinite is altogether incomprehensible to knowledge; rather it takes it hypothetically and uses only the finite for demonstration; that is, it assumes the infinite not for the sake of the infinite, but for the sake of the finite. If our imagination could see that the given point does not lie on the extension of the finite line and is so separated from it that no part of the line could underlie the point, the demonstration would no longer need the infinite. It is therefore that it may use the finite line without the risk of refutation or doubt that it posit the infinite, relying on the boundlessness of imagination as the source which generates it[10].

[9] Proclus, *A Commentary* ..., *op. cit.*, p. 148.

[10] Proclus, *op. cit*., pp. 221–223. Cf. Also the Greek text of Friedlein: "οὐδὲν ἔτι τοῦ ἀπείρου δεήσεται. ἵν᾽ οὖν τῇ πεπερασμένῃ χρωμένη ἀνελέγκτως χρήσηται καὶ ἀναμφισβητήτως, εἶναι τὸ ἄπειρον ὑποτίθεται τῇ τῆς φαντασίας ἀοριστίᾳ τῆς τοῦ ἀπείρον γενέσεως ὑποβάθρᾳ χρωμένε". (Cf. Proclus, *Procli Diadochi in Primum* …, *op.cit*, p. 286, 7–11).

Earlier, (in 158,6), Proclus explains that εἰς ἄπειρον means ἐπ᾽ἄπειρον. Therefore, "infinite straight line" in I. 12 means "finite line of an indefinite longitude sufficient for the accomplishment of the given construction", i.e. it is an undefined finite line without exact boundaries. Cf. for instance: "[…] for when Euclid says that the line is limited by points, he is clearly making the line as such unlimited, as not having any limit because of its own forthgoing."[11]. Such ideas are not understandable at present because for us an "infinite line" is always "actually infinite". Proclus explains this point later:

> They cannot be limits of the infinite line, nor of every finite line. For there is a line which is finite but does not have points as its limits. The circle is such a line, bending back upon itself and making no use of limits as does the straight line. Such also is the ellipse. Perhaps, then we should consider the line only insofar as it is a line[12].

G. L. Morrow adds a comment: "This is a strange phrase, for it contradicts the statement (101, 7) that the line 'as such' is unlimited. The point of this paragraph is only to show that, *if* a line has limits, those limits are points"[13].

Euclid also uses lines that are undefined in one direction; cf. theorem I. 22 and Proclus' commentary "for in the construction he says *given a straight line limited in one direction but unlimited in the other*"[14].

Even if Euclid speaks about infinite or indefinite extension of a line, the context of this expression determines that the line under consideration is finite though "undefined" in the above sense. For instance, in the proof of I. 44 we have: "and straight lines produced indefinitely from angles less than two right angles meet; therefore *HB*, *FE*, when produced, will meet." (I. 44,19–21). One can produce lines further (Postulate I. 2), but for the sake of the proof, it is necessary to prolong them only "so far" as they will meet, and it is impossible to determine exactly "how far". "Infinite lines" play the role of some "geometrical variables" used in the constructions.

This point is also visible in the two Geminus' classifications of lines extant in Proclus' *Commentary*[15]. In the first classification, we have lines which do not form a figure, indeterminate (*ἀόριστοι*) lines, and lines "ex-

[11] Cf. Proclus, *A Commentary …*, *op. cit.*, p. 82. Cf. also Proclus, *Procli Diadochi in Primum …*, *op. cit.*, p. 101, 7.

[12] Cf. Proclus, *A Commentary …*, *op. cit.*, p. 84. Cf. also Proclus, *Procli Diadochi in Primum …*, *op. cit.*, p. 103, 7.

[13] Cf. Footnote 29 in Proclus, *A Commentary …*, *op. cit.*, p. 84.

[14] Cf. also Proclus' remarks on I. 11.

[15] Cf. *ibidem*, pp. 90–91 (or in Greek, cf. Proclus, *Procli Diadochi in Primum …*, *op. cit.*, the first: p. 111, 1–9, the second: pp. 111, 9–20; 112, 16–18) and T.L. Heath, *The thirteen books of Euclid's «Elements'*, vol. I, pp. 160–162.

tending without limit" (*ἐπ᾽ἄπειρον ἐκβαλλόμεναι*). As an example of lines extending without limits, Proclus gives the asymptotic lines (*ἀσύμπτωτοι*), which "never meet, however they are produced". Thus, these lines are also extendable lines without defined limits, i.e. points, and they are different from the lines used by Euclid. I explain below that infinite lines were known in Antiquity also in the times of Euclid (and even before him). However, in fact, they are totally absent from *Elements* and they were methodically removed from Euclid's geometry by Euclid and other ancient mathematicians (cf. also below the note about the parallels).

In the second classification of lines, we have also indeterminate lines. However they create a different group from the lines which extend without limits (*ἡ ἐπ᾽ἄπειρον ἐκβαλλομένη*). These straight lines are ἀόριστοι, i.e. without limiting points, and they are different from the lines extending without limits.

Proclus explains in many places that the lines used by Euclid are not actually infinite[16]. The most important explanations are found on pages (in the English edition of *Commentary*): 144–146, 155, 175, 216–217 in the context of the bisection of an infinite line and commensurability and incommensurability, 333[17]. Also, in some alternative proofs to Euclid's proofs, Proclus extends lines; cf. pp. 171, 187, 258, etc.

Proclus explains theorem I. 11 as follows:

> Whether we take the straight line as limited in both directions, or unlimited in both, or unlimited in one and limited in the other, with the point lying on it, the construction with which our geometer solves this problem succeeds. For even if the given point lies on the extremity of the line, we can produce the same construction by extending the straight line. Clearly the point here is given in position, and in position only as lying on the straight line; but the straight line is given only in kind, for its length, *ratio* [it indicates on the basic line and that the ancients do not consider "every possible" line for us — Z.K.], and position are not determined[18].
>
> But we must understand that the character of being produced indefinitely does not belong to all lines. It belongs neither to the circular nor to the cissoid, nor to any of the figure-describing lines, nor even to all those that do not make figures. For not even the monostrophic spiral

[16] Proclus explains also that "The Pythagoreans consider quantity and magnitude not in their generality, however, only as finite in each case". (Proclus, *op. cit.*, p. 30).

[17] This point is essential for Book X: "The straight line must be finite, for it is not possible to apply an area to an infinite line; so in saying [cf. Euclid I. 44 – Z.K.] that we are to apply an area to 'a given straight line', he makes it clear that the line is necessarily finite".

[18] Proclus, *A Commentary ...*, *op. ci*t., p. 218.

> can be produced indefinitely, *since it has its existence between two points* [i.e. is defined and determined strictly — Z.K.]; nor can any other of the lines so generated. Nor it is possible to join every point with every other point by every line, for not every line can exist between all points [for us it is possible — Z.K.][19].

One more example of a conceptual difference between ancient and modern geometry is the concept of angle. I will not explain this in detail, and will indicate only one difference, i.e. the sides of an angle are always finite lines or lines that are indeterminate — in the sense explained above — lines (as in I. 23)[20]. However, an angle is one and the same for many different sides as one and the same inclination of two lines[21]. Therefore, there is only one right angle for Euclid and Proclus as being the same inclination between an infinite number of different finite lines[22]. Proclus extends the sides of an angle in some proofs[23]. The finite nature of an angle's sides is in some places the sine qua non condition for the understanding of the text[24].

The absence of the actually infinite line is visible particularly in the theory of the parallel lines. This point also indicates some differences between Euclid's and Proclus' times.

[19] *Ibidem*, p. 147.

[20] Cf. also Proclus' phrase "angles BAC and DCA are defined by points F and G" on indefinite lines AB and CD, Proclus, *A Commentary …*, *op. cit.*, p. 292.

[21] There are many other differences, and there were at least three different approaches to this concept; cf. Proclus, *op. cit.*, pp. 98–104. For Apollonius, the angle is determined by only one broken line (cf. *Ibidem*, p. 99). For Euclid, a plane angle (in ancient mathematics, "angle" has a much broader meaning and contains also angles determined not only by straight lines, as, for instance, horned angles) is the inclination to one another of two lines; cf. complete Definition I. 9. Every angle was less than two right angles and other modern angles were not considered as angles (cf. *Ibidem*, p. 257 and footnote 159), for example some exterior angles; cf. *Ibidem*, pp. 131, 239 ("For the triangle as such never has an exterior angle."), 257. In the consequence, there was a four-sided-triangle; cf. Proclus, p. 130-131, 257 (or in Greek, Proclus, *Procli Diadochi in Primum …*, *op.cit*, pp. 165–166, 259). For us, a triangle is mainly a three-sided polygon and is determined by three sides; for them — by three angles, and the concept of an angle was different. The finiteness of the sides of an angle results also in different conditions of the equality of angles. Cf. "Every straight line coincides with every other, and in the case of equal lines their extremities [i.e. the lines are finite — Z.K.] also coincide. A rectilinear angle is said to be equal to the rectilinear angle when, if one of the sides containing it is placed upon one of the sides containing the other, the second side of the first coincides with the second side of the other. […] since it is possible for angles to be equal without having their sides congruent. [cf. some lunular and rectilinear angles]", (*Ibidem*, p. 185–186).

[22] Cf. Proclus, *A Commentary …*, *op. ci*t., pp. 99, 282 (explanation of the phrase "at right angles", see Proclus, *Procli Diadochi in Primum …*, *op. cit.*, p. 282, 2).

[23] Proclus, *A Commentary …*, *op. ci*t., pp.. 266, 279.

[24] Cf. *ibidem*, p. 149 ("Let us imagine two equal lines *AB* and *BC* making a right angle at *B* …").

In the famous fifth postulate (Postulate I. 5), Euclid speaks about the indefinite prolongation of lines: "That, if a straight line falling on two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles"[25]. How easy it is to miss this point can be seen in Fitzpatrick's new translation:

> And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then, being produced to infinity, the two (other) straight-lines meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side)[26].

The same point is missed in Fitzpatrick's translation of definition I. 23 of the parallels. Heath's translation is as follows: "Parallel straight lines are straight lines, which being in the same plane and being produced indefinitely in both directions [so, these lines previously are finite and parallel! — Z.K.], do not meet one another in either direction"[27]. Heath explains:

> *Εἰς ἄπειρον* cannot be translated 'to infinity' because these words might seem to suggest a *region* or *place* infinitely distant, whereas *εἰς ἄπειρον*, which seem to be used indifferently with *επ΄ἄπειρον*, is adverbial, meaning 'without limit', i.e. 'indefinitely'. Thus the expression is used of a magnitude being 'infinitely divisible', or of a series of terms extending without limit. *In both directions*, *ἐφ΄ἑκάτερα τὰ μέρη*, literally 'towards both the parts' where 'parts' must be used in the sense of 'regions'[28].

One can compare some proofs and theorems in Book I to see that Euclid uses finite parallel lines (cf. I. 33)[29] and even prolongs them (cf. the proof of I. 31). In general: in every place in which we have an "infinite line"

[25] "Καὶ ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ελάσσονας ποιῇ, ἐκβαλλομένας τὰς δύο εὐθείας ἐπ ᾽ἄπειρον συμπίπτειν, ἐφ᾽ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες". Cf. T. L. Heath, *The thirteen books* …, vol. I, p. 202.

[26] Cf. Euclid, *Euclid's Elements* …., R. Fitzpatrick (ed.), *op. cit.*

[27] Cf. T.L. Heath, *The thirteen books* …, *op. cit.*, vol. I, p. 190.

[28] *Ibidem*. Cf. also Def. I. 23 in Greek: "παράλληλοί εἰσιν εὐθεῖαι, αἵτινες ἐν τῷ αὐτῷ ἐπιπέδῳ ὗσαι καὶ ἐκβαλλόμεναι εἰς ἄπειρον ἐφ᾽ἑκάτερα τὰ μέρη ἐπὶ μηδέτερα συμπίπτουσιν ἀλλήλαις" (J. L. Heiberg, *Euclidis Elementa*, *op. cit.*, vol. I). Cf. Thuc. II. 96

[29] Heath adds the phrase "at the extremities which are", that is justified on the grounds of the following proof in *Elements*.

(I. 22[30], I. 29[31]), Euclid speaks about an indefinite though finite, extension of the line and the actual infinity is inessential to the proof. Theorem I. 10 shows how to divide every line but an actually infinite line has no middle; cf. Aristotle *Topics* 148b.

Euclid gives two f i n i t e criteria for lines that are sufficient for them to be parallel: I. 27 and I. 28. For example, in proof I. 27, he states that: "I say that *AB* is parallel to *CD*. For, if not, *AB*, *CD* when produced will meet either in the direction of *B*, *D* or towards *A*, *C*. Let them be produced and meet in the direction of *B*, *D*, at *G*", *etc*. The same is in I. 37, I. 38, I. 40, *etc*.

I recommend that the reader check the history of Euclid's fifth postulate, and see that, in almost every ancient and medieval Arabic commentary or attempt to prove the postulate, only finite parallels and, in almost every proof, the extensions of finite lines are used[32].

As I have already mentioned, there are differences between Proclus and Euclid, and thus infinite lines were known in Antiquity. Proclus, explaining theorem I. 30, i.e. that straight lines parallel to the same straight line are also parallel to one another, says (contrary to Euclid):

> For we must think of parallel lines as produced indefinitely, and *AH* when produced coincides with *HB*; it is therefore the same as it, and not another line. Therefore all the parts of a parallel line are themselves parallel to the straight line to which it is parallel, both to the whole of

[30] "Ἐκκείσθω τις εὐθεῖα ἡ ΔΕ πεπερασμένη μὲν κατὰ τὸ Δ ἄπειρος δὲ κατὰ τὸ Ε [...]"; (cf. J. L. Heiberg, *Euclidis Elementa*, *op. cit.*, vol. I, pp. 11–12). In reality, *DE* is "as long" as it is needed for the construction.

[31] Cf. Heiberg's text of I. 29, (J. L. Heiberg, *Euclidis Elementa*, *op. cit.*, vol. I): "αἱ δὲ ἀπ΄ἐλασσόνων ἢ δύο ὀρθῶν εκβαλλόμεναι εἰς ἄπειρον συμπίπτουσιν· αἱ ἄρα ΑΒ, ΓΔ ἐκβαλλόμεναι εἰς ἄπειρον συμπεσοῦνται· οὐ συμπίπτουσι δὲ διὰ τὸ παραλλήλους αὐτας ὑποκεῖσθαι". ("But straight lines produced indefinitely from angles less than two right angles meet, therefore AB, CD, if produced indefinitely, will meet; but they don't meet, because they are by hypothesis parallel.", T.L. Heath, *The thirteen books ...*, *op. cit.*, vol. I).

[32] Cf. for instance B.A. Rosenfeld, *A history of non-Euclidean geometry: Evolution of the concept of a geometric space*, Studies in the History of Mathematics and Physical Sciences 12, New York/Berlin et al. 1988, chapter 2, pp. 35–96. Rosenfeld quotes Posidonius' (ca. 135–51 B.C.) definition of parallels: "We define parallel lines as lines that are in the same plane, and such that if they are indefinitely produced in both directions, the distance between them stays the same; the distance is the shortest line joining them, the same is said of other distances" (p. 43). Typical examples are the proofs of al-Hanafi ("the line *AC*, if produced, will not meet the line *ED*", p. 45), and of al-Jawhari ("If one extends two lines from a line in the direction of angles less than two right angles, then they intersect on that side", p. 49). We can observe a gradual change of meaning, cf. especially Khayyam (X/XI A.D.; "A straight line can be produced to infinity", p. 38) and al-Nairizi (IX/X A.D.).

> it and to its parts. Thus *AH* is parallel to *KD* and *HB* to *CK*; for when produced indefinitely they remain nonsecant[33].

He also proves the uniqueness of the parallel line through the given point and parallel to the given line: "From the same point two perpendiculars cannot be drawn to the same straight line, nor through the same point can two parallels be drawn to the same straight lines"[34].

However, Proclus' statement is in contradiction with Euclid's explicit proof from *Elements*. Proclus mixes the meanings of the actually infinite line and the undefined or undetermined line: there is really only one parallel and undefined line; i.e. a line without the limited points, and many finite parallel lines. He is also not firm in the above statement, and contradicts himself in many places; cf. *ibidem*, p. 303 ("For a parallelogram is formed by the equal and parallel lines")[35] or p. 312.

To finish off the problem of the finiteness or infiniteness of the parallels, I will quote the older commentary of Theon: "Straight lines are parallel when, prolonged to infinity on the same plane, they do not meet and always maintain the same distance between them"[36]. From Theon's commentary, it is known that light rays, shadows and every line in astronomy were considered finite[37]. Astronomers also used Postulate I. 2 and the operation of the prolongation of (finite) lines[38].

The finitism of ancient mathematics is also visible in cases other than that of straight lines, such as the cissoid and some spirals. For instance, Heath comments: "But what is surprising is that Proclus seems to have no idea of the curve passing outside the circle and having an asymptote, for he several times speaks of it as a closed curve (forming a figure and including an area): p. 152, 7. [...] Even more peculiar is Proclus' view of the 'single-turn spiral'"[39]. Even the position of Archimedes is finitistic. However, he considers more than one revolution of spirals; cf. his *On Spirals* and his explanations given to Disitheus and Definitions. Archimedes also uses other turns of spirals in

[33] Proclus, *A Commentary ...*, *op. cit.*, p. 294.

[34] *Ibidem*, p. 296.

[35] Cf. I. 33, to which the comment refers, where Euclid speaks about finite parallel lines.

[36] Theon of Smyrna, *Mathematics useful for understanding Plato by Theon of Smyrna, Platonic Philosopher*, Ch. Toulis (ed.), translated by R. Lawlor and D. Lawlor with an appendix of notes by J. Depuis, San Diego 1979, p. 74. (Cf. also Theon of Smyrna, *Theonis Smyrnaei philosophi platonici expositio rerum mathematicarum ad legendum Platonem utilium*, E. Hiller (ed.), Lipsiae 1878.)

[37] Theon of Smyrna, *op. cit.*, pp. 126–128.

[38] *Ibidem*, p. 127.

[39] T.L. Heath, *The thirteen books...*, vol. I, pp. 164–165.

some proofs, e.g. Theorems 16, 17, *etc.*[40]. Every spiral is only a finite line with exact extremities.

The main exception to this finitism is presented by Apollonius of Perga. In contradiction to Euclid, Archimedes and others who, in their definitions and propositions, use only determined finite cones, Apollonius defines an indefinite conic line and conic surface, and discerns these objects from every given cone:

> If a straight line indefinite in length, and passing through a fixed point, be made to move round the circumference of a circle which is not in the same plane with the point, so as to pass successively through every point of that circumference, the moving straight line will trace out the surface of a double cone, or two similar cones lying in opposite directions and meeting in the fixed point, which is the apex of each cone[41].

[40] Cf. for instance: "If a straight line of which one extremity remains fixed be made to revolve at a uniform rate in a plane until it returns to the position from which it started, and if, at the same time as the straight line revolves, a point moves at a uniform rate along the straight line, starting from the fixed extremity, the point will describe a spiral in the plane"; cf. T. L. Heath, *The Works of Archimedes Edited in Modern Notation with Introductory Chapters by T. L. Heath with a Supplement 'The Method of Archimedes' Recently Discovered by Heiberg*, Dover Publications, Inc., New York (1912). Retrieved 2009, from http://www.wilbourhall.org/, p. 154 (*On spirals*). Obviously, the famous Archimedes' lemma assumes also only finite lines and areas ("if there be (two) unequal lines or (two) unequal areas, the excess by which the greater exceeds the less can, by being [continually] added to itself, be made to exceed any given magnitude among those which are comparable with [it and with] one another", p. 155. Cf. also *The quadrature of the parabola*, *ibidem*, p. 234. The same finitistic approach is presented by Archimedes to the conoids and spheroids. He defines them as some sections of the given and determined cone. Archimedes uses also the prolongation of lines (for example: "if from [one magnitude another magnitude be subtracted which has not the same center of gravity, the center of gravity of the remainder is found by] producing [the straight line joining the centers of gravity of the whole magnitude and of the subtracted part in the direction of the center of gravity of the whole] and cutting off from it a length which has to the distance between the said centers of gravity the ratio which the weight of the subtracted magnitude has to the weight of the remainder"; *The method of Archimedes*, *ibidem*, p. 14, *On the equilibrium of planes*, Proposition 8, *ibidem*, p. 194.

[41] Cf. T.L. Heath, *Apollonius of Perga: Treatise on Conic Sections with Introductions Including an Essay on Earlier History on the Subject*, Cambridge 1896, available also on the Internet: retrieved 2009, from http://www.wilbourhall.org/, p. 1. Cf. also translation from Rosenberg's edition (available on the Internet also): "If from a point a straight line is joined to the circumference of a circle which is not in the same plane with the point, and the line is produced in both directions, and if, with the point remaining fixed, the straight line being rotated about the circumference of the circle returns to the same place from which it began, then the generated surface composed of the two surfaces lying vertically opposite one another, each of which increases indefinitely as the generating straight line is produced indefinitely, I call a conic surface, and I call the fixed point the vertex, and the

Based on Apollonius' *Conics*, it is necessary to differentiate between the actually infinite line or surface and indefinited in length, i.e. an unlimited line and surface respectively. Apollonius uses the prolongation of lines, surfaces and parallel lines in some proofs. For example:

> Again, if *PM* be the diameter of a section made by a plane cutting the circular base in the straight line *DME* perpendicular to *BC* , and if *PM* be in such a direction that it does not meet *AC* though produced to infinity, i.e. if *PM* be either parallel to *AC*, or makes with *PB* an angle less than the angle *BAC* and therefore meets *CA* produced beyond the apex of the cone, the section made by the said plane extends to infinity. For, if we take any point *V* on *PM* produced and draw through it *HK* parallel to *BC*, and *QQ′* parallel to *DE*, the plane through *HK*, *QQ′* is parallel to that through *DE*, *BC*, i.e. to the base. Therefore the section *HQKQ′* is a circle. And *D*, *E*, *Q*, *Q′* are on the surface of the cone and are also on the cutting plane. Therefore the section *DPE* extends to the circle *HQK*, and in like manner to the circular section through any point on *PM* produced, and therefore to any distance from *P*.

or

> in the case where the diameter *PM* meets *CA* produced beyond the apex of the cone, both *VK*, *HV* increase together as *V* moves away from *P*. Thus *QV* increases indefinitely as the section increases to infinity., *etc.*[42].

Heath explains Apollonius' terms: *ἄπειρος* means "unlimited" or "infinite" and *εἰς ἄπειρον αὐξάνεσθαι* — "to increase without limit" or "to increase indefinitely"[43].

Apollonius' use of undetermined lines and surfaces, i.e. objects without boundaries, is really different in comparison with the works on conic sections by Euclid and Archimedes; however, it is impractical to give more evidence for this difference here[44].

straight line drawn from the vertex to the center of the circle I call the axis. Cf. Greek text in Apollonius of Perga, *Apollonii Pergaei Que Graece Extant Cum Commentariis Antiquis. Edidit Et Latine Interpretatus Est J.L. Heiberg*, vols. I, II, J. L. Heiberg (ed.), Lipsiae 1891. Retrieved 2008, from http://www.wilbourhall.org/.

[42] T.L. Heath, *Apollonius of Perga: Treatise on Conic…*, *op. cit.*, pp. 5–6.

[43] *Ibidem*, p. clxx.

[44] Heath explains also that "what we call asymptotes (αἱ ἀσύμπτωτοι in Apollonius) are in Archimedes the lines (approaching) nearest to the section of the obtuse-angled cone *αἱ ἔγγιστα τᾶς τοῦ ἀμβλυγωνίου κώνου τομᾶς*; cf. T.L. Heath, *The Works of Archimedes…*, p. vclxviii.

Now, it is possible to recognize the shift of meaning of Euclid's fifth postulate in modern times. From the viewpoint of the contemporary intuitive infinite model for Euclidean geometry, there are some lacunae in *Elements*. For instance, Hartshorne, Heath *et al.* indicate that it is possible and necessary to prove the uniqueness of the parallel line through the given point and parallel to the given line. In modern geometry, this statement is an axiom, the so-called Playfair's axiom, and it is possible to prove it using I. 29. It is also known that in the modern infinite model, it is possible to prove without Postulate I. 5 that "if a straight line falling on two straight lines makes the angles on the same side of this line equal to two right angles, the straight lines are non secant". However, to do this, it is necessary to assume that every straight line divides the plane on which it is situated into two disjointed areas.[45] Cf. also Leibniz's definitions of a straight line as "one which divides a plane into two halves identical in all but position" and of a plane as "that surface which divides space into two congruent parts" (T. L. Heath, *The thirteen books…*, vol. I, pp. 169 and 176 respectively). From the book of Hartshorne[46], the reader can recognize how — from the technical point of view — it is possible to change the meaning of ancient geometry and to treat it as "immersed" in infinite objects. However, Harsthorne or, for example, Newton and many others writing books about "Euclidean" geometry were not aware that their absolute space, line and surface are absent from *Elements*.

The same situation as with lines occurs with plane surfaces and with surfaces in general. Euclid states that "the extremities of a surface are lines" (Def. I. 6). From Heron's classification of surfaces we know that there was an analogue to the operation of the line's prolongation, i.e. the extension of surfaces. For instance, composite surfaces are "those which when produced cut one another", or incomposite surfaces — "those which, when produced, coalesce with themselves".[47] The classification of surfaces was similar to Geminu's classification of lines. Also, Definition 8 of parallel planes in Book XI is similar to that of lines: "Parallel planes are those which do not meet". Heath explains that: "Heron has the same definition… (Def. 115). The Greek word which is translated 'which do not meet' is ἀσύμπτωτα, the term which has been adopted for the asymptotes of a curve"[48].

[45] Cf. E.W. Beth, *Mathematical thought. An introduction to the philosophy of mathematics*, Dordrecht 1965, p. 8.

[46] Cf. R. Hartshorne, *Geometry: Euclid and Beyond*, New York *et al.* 2000.

[47] Cf. Definition 74 [in:] J.L. Heiberg, *Euclidis Elementa*, vol. I: *ὅσαι ἐκβαλλόμεναι αὐταὶ καθ᾽ ἑαυτῶν πίπτουσιν, etc.*

[48] Cf. T.L. Heath, *The thirteen books…*, vol. III, p. 265. Cf. also Ptolemy's proof "that straight lines produced from angles less than two right angles meet in the direction in which lie

Proclus explains in connection with parallel lines (I. 27) that: "Thus it is presupposed that everything that we write about in plane geometry we imagine as lying in one and the same plane"[49]. For us, this plane is actually infinite, for the ancients — an indefinite plane surface without boundaries, i.e. without any indicated lines limiting this surface. Therefore, in every possible geometrical place, it is impossible to indicate that the plane finishes in this or that place. Such an indefinite plane was a precondition to performing a given plane construction determined by the permissible operations such as the prolongation of lines. Therefore, such a plane extends during the constructions and, without any construction, it is impossible to say that there is any place determined and defined as already present there, outside the figure already constructed.

There are numerous known alternative proofs of some of Euclid's theorems in which their authors tried to avoid any acceptance of the place in geometry prior to construction. Cf. for example Heron's proofs of I. 20, or his mentions about the proofs of I. 16, I. 48, or Proclus' proofs of I. 5, I. 17, I. 32 or I. 9. Proclus explains that "Εἰ δὲ λέγοι τις τόπον μὴ εἰδέναι[50] [...], λέγει οὖν τις ὅτι οὐκ ἔστιν τόπος"[51]. The following commentary of Proclus concerning theorem I. 16 (i.e. in any triangle, if one of the sides be produced, the exterior angle is greater than either the interior and opposite angles) is in the same spirit:

> Some persons have cited this enunciation elliptically, without 'if one of the sides is produced', and thereby have given occasion — perhaps to others and certainly to Phillippus, as Heron the engineer tells us — to criticize it. For the triangle as such never has an exterior angle[52].

There are also known alternative proofs of the theorems trying to avoid the exterior prolongation of a line and presented by "the followers of Heron and Porphyry"[53]. The same situation prevails with solids: "A solid is that which has length, breadth, and depth" (Def. XI. 1) and "An extremity of a solid is a surface" (Def. XI. 2). Proclus explains:

the angles less than two right angles" (Proclus, *op. cit.*), to see that asymptotic lines where also indefinite but not actually infinite.

[49] Proclus, *A Commentary...*, p. 278.

[50] *Idem*, *Procli Diadochi in Primum...*, p. 275, 7.

[51] *Ibidem*, p. 289, 18–19, cf. *Ibidem* 225, 16; *idem*, *A Commentary...*, pp. 177, 214, 226, 340–341. For example: "Someone may say there is no room on the other side of the line, but only on the side on which C lies." (p. 226).

[52] *Idem*, *A Commentary...*, pp. 238–239

[53] Cf. *Ibidem*, p. 252.

> In order that an object in three dimensions may not stretch to an infinite size in our thought or perception, it is limited on all sides by planes; and so that the plane may not slip into boundlessness, the line comes to be in it to contain and define it; and the point does the same thing for the line, these simples existing for the compounds. (*Ibidem*, p. 71).

The commentary of Pappus on Book X of *Elements* explains that:

> If, then, the reason be demanded why a minimum but not a maximum is found in the case of a discrete quantity, whereas in the case of a continuous quantity a maximum but not a minimum is found, you should reply that such things as these are distinguished from one-another only by reason of their homogeneity with the finite or the infinite […][54].

The conviction of the finiteness of geometrical magnitudes is presented very often in ancient philosophy. Aristotle's concepts of actual and potential infinity are well known and, in the present paper, it is impossible to give a systematic account of his views on other ancient views on infinity in mathematics and on nature that are evident in his writings. I can indicate only several important points and quote some papers in support of the above.

> The infinite turns out to be the contrary of what it is said to be. It is not what has nothing outside it that is infinite, but what always has something outside it. This is indicated by the fact that rings also that have no bezel are described as 'endless', because it is always possible to take a part which is outside a given part. The description depends on a certain similarity, but it is not true in the full sense of the word. This condition alone is not sufficient: it is necessary also that the next part which is taken should never be the same. In the circle, the latter condition is not satisfied: it is only the adjacent part from which the new part is different.
>
> Our definition then is as follows:
>
> A quantity is infinite if it is such that we can always take a part outside what has been already taken[55].
>
> If 'bounded by a surface' is the definition of body there cannot be an infinite body either intelligible or sensible. […] 'Body' is what

[54] Cf. W. Thomson, G. Junge, *The Commentary of Pappus on Book X of Euclid's Elements*. Arabic Text and Translation by William Thomson with Introductory Remarks, Notes, and a Glossary of Technical Terms by Gustav Junge and William Thomson, vol. VIII, Cambridge/London 1930, part I, par. 3, p. 65(3), such statements are many, also in Aristotle.

[55] Cf. Aristotle, *Physics* 206b 33–207a 1; cf. Book III, Part 6, [in:] R.P. Hardie, R.K. Gaye, *Physics by Aristotle*, eBooksAdelaide. Retrieved 2007, from: http://classics.mit.edu/ Aristotle/physics.3.iii.html.

> has extension in all directions and the infinite is what is boundlessly extended, so that the infinite body would be extended in all directions ad infinitum[56].

The problem of infinity is discussed by Aristotle mainly in his *Physics* (especially in Book III and IV), *Metaphysics* (in many places, cf. for example books XIII and XIV), *De caelo* (cf. for instance 271b–276a) and *De generatione et corruptione* (for example 332b–333a).

> (4) Again, as a line which has a limit cannot be infinite, or, if it is infinite, is so only in length, so a surface cannot be infinite in that respect in which it has a limit; or, indeed, if it is completely determinate, in any respect whatever. Whether it be a square or a circle or a sphere, it cannot be infinite, any more than a foot-rule can. There is then no such thing as an infinite sphere or square or circle, and where there is no circle there can be no circular movement, and similarly where there is no infinite at all there can be no infinite movement; and from this it follows that, an infinite circle being itself an impossibility, there can be no circular motion of an infinite body. (5) Again, take a center *C*, an infinite line, *AB*, another infinite line at right angles to it, *E*, and a moving radius, *CD*. *CD* will never cease contact with *E*, but the position will always be something like *CE*, *CD* cutting *E* at *F*. The infinite line, therefore, refuses to complete the circle[57].

Therefore, the concept of actual infinity was known in Antiquity. However, it was removed from mathematics. One more reason for this elimination was the indeterminate character of infinite objects from the viewpoint of the ancient theories of proportion: "But there is no proportion between the infinite and the finite: proportion can only hold between a less and a greater finite time"[58]. And, the last quotation:

> With magnitudes the contrary holds. What is continuous is divided ad infinitum, but there is no infinite in the direction of increase. For the size which it can potentially be, it can also actually be. [...]
>
> Our account does not rob the mathematicians of their science, by disproving the actual existence of the infinite in the direction of increase, in the sense of the untraversable. In point of fact they do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish. It is possible to have divided in the same ratio as the largest quantity another

[56] *Ibidem*, Book III, Part 5.

[57] Cf. Aristotle, *De caelo*, book I, part 5, transl. J. L. Stocks; retrieved 2004, from: http://classics.mit.edu/Aristotle/heavens.html.

[58] Cf. *ibidem*, Book I, part VI, J. L. See also various places in *Physics*, especially in book IV.

> magnitude of any size you like. Hence, for the purposes of proof, it will make no difference to them to have such an infinite instead, while its existence will be in the sphere of real magnitudes[59].

Modern physics and calculus emerged when the concepts of actual infinity were introduced into Euclidean geometry. For Newton, it was necessary that a body could move along an infinite straight line in the absence of any other forces. Therefore, the infinite mathematical objects have become necessary and simply useful[60].

2. The One and the Dyad

In *Elements*, every specific geometrical and mathematical object — a triangle, a line, a square, a number… — is composed of that which is unlimited, undetermined, undefined and that which delimits, defines and bounds such objects. The line is limited by points, a surface by lines, a solid by surfaces etc. The line given without ending points is undetermined in length. However, this line is determined as being a line, and not, for example, a surface.

Instead of infinite absolute space in ancient geometry, a totally undefined and unlimited "spatial place" existed. Such an undefined place possesses all the properties of what was called the Dyad in Plato's philosophy. On the other hand, all limits are connected with the other highest principle: the One.

The presence of the modern infinite model for Euclidean geometry makes it impossible to notice the purely mathematical content of the One and the Dyad of Plato's protology. Therefore, usually, such repeals are treated in modern times as purely philosophical and, in fact, external to the ancient mathematical theories, as numerical mysticism, theurgy, *etc*., are external to arithmetic.

From the previous section the reader can see that the treatment of geometrical objects as composed of the undefined Dyad and some limiting factors, are internal facts resulting in the specific properties of mathematical

[59] Cf. R.P. Hardie, R.K. Gaye, *Physics by Aristotle*, Book III, Part 7.

[60] Aristotle, in contradiction to Newton, states that: "Consequently there cannot be a continuous rectilinear motion that is eternal. […] The straight line traversed in rectilinear motion cannot be infinite: for there is no such thing as an infinite straight line; and even if there were, it would not be traversed by anything in motion: for the impossible does not happen and it is impossible to traverse an infinite distance. […] In rectilinear motion we have a definite starting-point, finishing-point, middle-point, which all have their place in it in such a way that there is a point from which that which is in motion can be said to start and a point at which it can be said to finish its course (for when anything is at the limits of its course, whether at the starting-point or at the finishing-point, it must be in a state of rest)", R.P. Hardie, R.K. Gaye, *Physics by Aristotle*, book VIII, Part 9. Cf. also Part 10.

theories. It is also a well-known fact that the conviction of the existence of such two mutually irreducible components of mathematical reality as well as of the whole of nature, (*physis*, *kosmos*) was frequent in ancient times; cf. the Pythagoreans, Plato, Neo-Pythagoreanism, Neo-Platonism, etc.

I have to explain here that I do accept the existence of the so-called *Plato's unwritten doctrine* and the new model of Plato's philosophy elaborated in the school of Tübingen, by G. Reale, H. Krämer *et al.*, simply because of the fact that this is congruent with the historical sources. However, I do not accept many specific statements, especially concerning ancient mathematics. I can see also that it is possible to unify the content of unwritten doctrine with many other views, as for instance those of H-G. Gadamer and K. M. Sayre views. I also gave a new interpretation of Plato's philosophy and of some dialogues in the general framework of the new model of Plato's philosophy; cf. the bibliographical section. I proposed a new interpretation of the concepts of number, ideal number, participation, ideas and mathematical objects[61]. All of these explain why Plato has accepted two different highest principles.

The connections between Plato's philosophy, ancient mathematics and the theory of incommensurable magnitudes are known. Moreover, this theme is present in many ancient commentaries of Plato's philosophy and Euclid's geometry (Proclus, Theon, Pappus, …). Furthermore, today, many authors are simply forced to study ancient mathematics in order to understand Plato's philosophy and protology (i.e. the science about the highest principles in Plato's philosophy); cf. for instance the great work by Konrad Gaiser[62].

It is impossible to explain the details of Plato's protology in the present paper. Thus, I simply assume some knowledge about the Dyad and the One on the part of the reader, and I stress their role for ancient mathematics in some sources. The list of *Testimonia Platonica* should be extended.

I begin with Proclus' commentary on Euclid's *Elements*. In many places, Proclus tells us words similar to the following[63].

> Since the line is the first thing to have parts and to be a whole, and since it is both monadic because unidimensional and dyadic because of its forthgoing — for if it is an infinite line it partakes of the indefinite dyad, and if it is finite it requires two limits, a whence and a whiter — for these reasons it is an imitation of wholeness and of that grade

[61] Cf. e.g. Z. Król, *Platon i podstawy matematyki współczesnej*; *idem*, *The implicit Logic in Plato's Parmenides*, „Filozofia Nauki", 81 (2013), pp. 121–135.

[62] Cf. K. Gaiser, *Platons Ungeschriebene Lehre. Studien zur systematischen und geschichtlichen Begründung der Wissenschaften in der Platonischen Schule*. Appendix: *Testimonia Platonica. Quellentexte zur Schule und mündlischen Lehre Platons*, 2. Aufl. Stuttgart 1968.

[63] Cf. Proclus, *A Commentary...*; in the text I give numbers of pages.

of being which is extended oneness and generates duality"[64]. […] Every compound gets its boundary from the simple, and every divisible thing from the indivisible [cf. "the limits of a line are points" and "a point is what has no part" — Z.K.]. The principles of mathematics provide images of these truths; for when Euclid says that the line is limited by points, he is clearly making the line as such unlimited, as not having any limit because of its own forthgoing. So, just as the dyad is bounded by the monad and, when controlled by it, sets a term to its own unchecked boldness, so also the line is bounded by points. And being dual in nature, when it participates in the point, which contains the idea of unity, it does so in the fashion of a dyad. Now in imagined and perceived objects the very points that are in the line limit it, but in the region of immaterial forms the partless idea of the point has prior existence. As it goes forth from that region, this very first of all ideas expands itself, moves, and flows towards infinity and, imitating the indefinite dyad, is mastered by its own principle, unified by it, and constrained on all sides. Thus it is at once unlimited and limited — in its own forthgoing unlimited, but limited by virtue of its participation in its limitlike cause[65].

According to Plato, the point, if we may to say so, appears to bear the likeness of the One, for the One also is without parts, as he has shown in the Parmenides. Since there are three hypostases below the One — namely, the Limit, the Unlimited, and the Mixed [cf. *Philebus* — Z.K.] — it is through them that the species of lines, angles, and figures come to be. Corresponding to the Limit are, in surfaces, the circular line, the angle bounded by circular lines, and the circle, and, in solids, the sphere. To the Unlimited corresponds the straight line in all three groups, for it is found in them all […]. And the mixtures in all of them correspond to the principle of the Mixed. For there are mixed lines, such as spirals; mixed angles, such as the semicircular and the horned angles; mixed figures, such as sections of plane figures and arches; and mixed solids, such as cones, cylinders, and the like. Hence the Limit, the Unlimited, and the Mixed are present in all of them. Aristotle's opinion is the same as Plato's […][66] .

But whence comes the idea of figure and from what sort of principles is it perfected? I answer, first, that it owes its being to the Limit and the Unlimited and the Mixture of the two. This is why it generates some kinds by virtue of the Limit, others by reason of the Unlimited, and others according to the Mixed[67].

[64] Proclus, *op. ci*t., p. 81; cf. also the next explanations of surface and solid.
[65] *Ibidem*, pp. 82–83.
[66] *Ibidem*, pp. 84–85.
[67] *Ibidem*, p. 115.

> And if the infinite straight line is a symbol of the whole world of becoming in infinite and indeterminate change and of matter itself which possesses no boundary nor shape, and if the point lying outside carries the likeness of partless being devoid of anything material, most certainly, then, the perpendicular dropped from above would be an imitation of life proceeding immaculately from the One and Indivisible into the world of generation[68].
>
> From this we can see how equality is a measure and a boundary of inequality as well. For even though the diminuation and increase of the obtuse and acute angles is indefinite and undetermined, yet this increase and diminuation are said to be limited by the right angle. [...] The dyad, which is in itself indefinite, is a paradigm of their indeterminateness. Here it seems we have a manifest of the forthgoing of the primary causes which stand as a single boundary line ever the same about the indefiniteness of generation[69].
>
> You see how the propositions that demonstrate equality of angles or sides suit both equilateral and isosceles triangles, and those that demonstrate inequality suit both scalene and isosceles. The reason is that some triangles are the product of equality alone, some of inequality alone, and some of both, having one character by virtue of equality and another because of inequality. And there are some beings that are akin to the Limit, others to the Unlimited, and others that are generated from both by the principle of the Mixed. Thus this triad of principles permeates everything: lines, angles, figures and among figures the three-sided, the four-sided, and all their successors. But the Limit in geometrical forms is sometimes manifested through likeness, sometimes through equality; the Unlimited sometimes through unlikeness, sometimes through inequality; and the Mixed sometimes arises out of likenesses and unlikenesses, and sometimes out of equalities and inequalities[70].

Thus, Proclus explains the technical construction of proofs and the structure of *Elements* using the highest principles of Plato.

> The author [i.e. Euclid — Z.K.] has constructed triangles and compared them with respect to their equality or inequality, establishing their existence by construction and their identity and differences by comparison. For existence involves three factors: Being, Sameness, and Otherness, both quantitative and qualitative, according to the individual characters of the subjects concerned. Thus through these

[68] *Ibidem*, p. 227.
[69] *Ibidem*, p. 229.
[70] *Ibidem*, pp. 245–246.

> propositions as likenesses he has shown us that everything is both identical with itself and other than itself because of the plurality it contains; that is, all are the same with one another and other than each other, for equality and inequality have been discovered to exist in each single triangle and in two or more[71].

He explains for instance, how to understand the terms like "in every…", for example:

> These being the two attributes indicated [i.e. "greater" and "less"], it is clear that, when the author of the *Elements* says 'in every triangle', he does not mean the equilateral triangle, but 'any triangle that has a greater and a lesser side'[72].

The influence of the Dyad (or the earlier Pythagorean Unlimited) and the One (or the Limit) is present in many specific mathematical methods, proofs and sources. I suggest that the reader see how it was important in, for instance, some technicalities of Archimedes' proofs. I will give one example of these. In Archimedes' *On conoids and spheroids*, in one of the proofs (Proposition 4), he says: "Then O [a circle — Z.K.] be equal to the area of the ellipse. For, if not, O must be either greater or less than the ellipse", and next, he considers two cases, i.e. O is greater, and O is less than the ellipse. Almost every important proof, can be seen in *On the sphere and cylinder* and many others. For us, it is an obvious matter, but the ancients were self-aware of the dyadic character of such matters. As I said in the introduction of this paper, it is necessary to re-read almost all the sources to see how important and ubiquitous the highest principles are in ancient mathematics. Even the Greek names of parabola, ellipse and hyperbola, and the method of the application of areas indicate these principles; cf. explanations given by Proclus in his *Commentary*[73]. The famous Archimedes' lemma is also a result of the reflection on this theme. Today, such places are merely "evident from the technical point of view", and their connection with the highest principles remains unrecognized.

I would like to add one comment here concerning Plato's highest principles, although it obviously does not exhaust the subject. Plato speaks about the indefinite Dyad of what is Great and Small[74], and he explains

[71] *Ibidem*, p. 275.

[72] *Ibidem*, p. 246.

[73] Cf. *Idem*, *A Commentary…*, pp. 332–333 (pp. 419,15–420,23 in Proclus, *Procli Diadochi in Primum…*).

[74] Cf. Aristotle, *Metaphysics* 1088a: "Those who regard the unequal as a unity, and the dyad as an indeterminate compound of great and small, hold theories which are very far from being probable or possible. For these terms represent affections and attributes, rather

how geometric spatial magnitudes are generated out of the Dyad from the species of the Great and the Small; cf. for instance the testimonies in Aristotle's *Metaphysica*: 1085a 7–14, 992a 10–18, 1088a 15–21, 1089b 9–14, 1090b 32–1091a 5, 1091a 32, 1091 b 35–1092a 1. It is also known that Plato identified space and matter; cf. *Timaeus* (for example 50b–c) and Aristotle's *Physica* (209b 11–17)[75]. This identification, similar to Descartes' view, makes it possible to create objects of sense experience from the primarily spatial Dyad; cf. *Philebus*. Therefore, the first function of the Dyad is to provide indeterminate, undefined spatiality. This undefined spatiality is not actually infinite (as Aristotle seems to suggest in some places), because it can be more or less. From *Timaeus* as well as from Aristotle, we know that there is no place beyond the world. However, the world is finite[76]. With undefined spatiality, the possibility of further unlimited divisions is connected.

The second function of the Dyad is connected with the mechanisms of the generation of numbers. I do not explain this subject, but numbers were not given and were created from the highest principles[77]. Aristotle speaks about *γένεσις τῶν ἀριθμῶν*. Cf. *Metaphysica* 987b 14– 988a 17. I give one example: "καὶ γὰρ τῶν ἀριθμον ἕκαστος ἕν τι. ἑνοποιὸν μὲν γάρ τι τὸ ἕν αὐτό, διχοποιὸς δὲ ἡ ἀόριστος δυάς., ἥτις διαίρεσις γένεσίς ἐστιν ἀριθμῶν", Alexander Aphrodisiensis *In Met. ad* 988a 1 [58,1 – 58,23, Hayduck[78]] (cf. also *ad* 987b 33 [55,17 – 57,34 Hayduck]). The connection of numbers with the Dyad is stated in many sources; cf. *Met.* 1089b 9–14 or Plutarch *Quest. Plat.* III[79], 1001, Theon (cf. Theon of Smyrna, *op.cit*, and

than substrates, of numbers and magnitudes — 'many' and 'few' applying to number, and 'great' and 'small' to magnitude — just as odd and even, smooth and rough, straight and crooked, are attributes".

[75] "This is why Plato in the *Timaeus* says that matter and space are the same; for the 'participant' and space are identical. (It is true, indeed, that the account he gives there of the 'participant' is different from what he says in his so-called 'unwritten teaching'. Nevertheless, he did identify place and space.)"; cf. *op. cit*., Book IV, part 2.

[76] As Sayre indicates, (cf. K.M. Sayre, *Plato's Late Ontology. A Riddle Resolved*, Princeton 1983, p. 248), the mysterious "third nature" from *Timaeus* is named in many different ways: *ὑποδοχὴν* (49a 6, 51a 6), *ἐκμαγεῖον* (50c 2), *δεχόμενον* (50d 3), *ἑδραν* (52b 1), *χώρα* (52d 3, 52b 1), *τιθήνην* (49a 7, 52d 5), *μητέρα* (51a 5), and *τροφὸν*. All of these terms indicate the connection with the functions of the Dyad and its role as the highest principle.

[77] For a brief account of the problem of the generation of numbers; cf. e.g. W.D. Ross, *Plato's theory of ideas*, Oxford 1961. On generation of numbers in Plato's *Parmenides*, cf. Z. Król, *The implicit Logic…*, pp. 121–135.

[78] Cf. A. Aphrodisiensis, *In Aristotelis Metaphysica commentaria*, [in:] M. Hayduck (ed.), *Commentaria in Aristotelem Graeca 1*, Berlin 1881.

[79] Cf. Plutarch, *Platonic Questions*, (a part of *Morals*) corrected and revised by W.W. Goodwin, retrieved 2013, from: http://www.platonic-philosophy.org/files/Plutarch%20-%20Platonic%20Questions%20%28selections%29.pdf.

D. Lawlor with an appendix of notes by J. Depuis, *op. cit.*, pp. 18, 21, 25), Nicomachus, Porphyry. Every series of numbers starts with "1" and has a different pattern of generation. The second function of the Dyad is duplication, re-production, multiplication. Therefore, the Dyad is also a pattern of the indefinite plurality.

On the other hand, the One is responsible for every identity, likeness, definiteness, finiteness, delimitation. The One "bounds" and gives limits to the Dyad; cf. also *Philebus* 16cff., 23cff. The One is the highest measure; cf. *Met.* 1087b 33–34. The One is different from the Dyad, and they are mutually irreducible (cf. *Parmenides* and, also, Theophrastus' *Metaphysics* IV 33, 11a 16 – 11b 12). From this point of view, we can understand why points were not parts of a continuum for the ancients; cf. for example Aristotle's *Physica* (231a–234a, *etc.*). The function of the One is visible in every mathematical definition.

> Each definition invariably brings with it a specific property and character in which all things that fall under the definition participate — as, for example, the definition of triangle, rectilinear figure, or of figure in general[80].

We can understand from the above that, in ancient mathematics in general, and geometry in particular, there is a duality of the indefinite Dyad of that which is more and less, and the bounding One, instead of infinite "Euclidean" space. Therefore, the comments from the two Prologues to Proclus' *Commentary* are not some "philosophical digressions", but create a real mathematical explanation.

> To find the principles of mathematical being as a whole, we must ascend to those all-pervading principles that generate everything from themselves: namely the Limit [*πέρας*] and the Unlimited [*ἄπειρον*, *ἀπειρία*]. For these, the two highest principles after the indescribable and utterly incomprehensible causation of the One, give rise to everything else, including mathematical beings. From these principles proceed all other things collectively and transcendentally, but as they come forth, they appear in appropriate divisions and take their place in an ordered procession, some coming into being first, other in the middle, and others at the end. […] Mathematicals are the offspring of the Limit and the Unlimited, but not of the primary principles alone, nor of the hidden intelligible causes, but also of secondary principles that proceed from them and, in cooperation with one another, suffice to generate the intermediate orders of things and the

[80] Proclus, *A Commentary…*, p. 28.

> variety that they display. This is why in these orders of being there are ratios proceeding to infinity, but controlled by the principle of the Limit. For number, beginning with unity, is capable of indefinite increase, yet any number you choose is finite; magnitudes likewise are divisible without end, yet the magnitudes distinguish from one another are all bounded, and the actual parts of the whole are limited. If there were no infinity, all magnitudes would be commensurable and there would be nothing inexpressible or irrational, features that are thought to distinguished geometry from arithmetic; nor could numbers exhibit the generative power of the monad, nor would they have in them all the ratios — such as multiple and superparticular — that are in things. For every number that we examine has a different ratio to unity and to the number just before it. And if the Limit were absent, there would be no commensurability or identity of ratios in mathematics, no similarity and equality of figures, nor anything else that belongs in the column of the better. There would not even be any sciences dealing with such matters, nor any fixed and precise concepts. Thus mathematics needs both these principles as do the other realms of being[81].

The non-hypothetical science about the highest principles is not part of mathematics because mathematics "takes its principles from the highest science and, holding them without demonstration, demonstrates their consequences"[82]. This is the main reason why there are numerous references only to the f u n c t i o n s of the One and the Dyad and only in the mathematical commentaries.

As an example, I quote some passages from Theon's commentary and indicate some places in Pappus.

> On the other hand the first idea of the odd is therefore the monad, also in the world, the quality of odd is attributed to that which is defined and well ordered. On the other hand, the first idea of the even is the indefinite dyad, which causes that which is indefinite, unknown and disorderly to be attributed to the quality of even, in the world also. This is why the dyad is called indefinite, because it is not defined, as is the Unit (or monad)[83].

> Unity is the principle of all things and the most dominant of all that is: all things emanate from it and it emanates from nothing[84].

[81] Proclus, *A Commentary...*, pp. 5–6.
[82] *Ibidem*, pp. 26–27.
[83] Theon of Smyrna, *op. cit.*, p. 15; Theon's words are in connection with Pythagoreanism.
[84] *Ibidem*, p. 66.

Theon also contends that every kind of proportion follows from the One and equality, and, that every kind of proportion is reducible to these principles[85]. Therefore, even in the theories of proportions we have a "constructive attitude"[86].

The term "Dyad" is not present in Pappus' commentary on Book X of *Elements*. However, the functions of the Dyad are frequent and essential for mathematical explanations. Pappus uses the functions of the Dyad in the explanations of every kind of manifestation of infinity in arithmetic and geometry. These functions are exactly the two functions above of the Dyad as described by Plato; cf. Part I of *Commentary*: paragraph 1, where Pappus takes into consideration also the One, writing about the definitional definiteness; paragraph 3, the most important testimony in Part I indicating *explicite* Plato; and paragraph 6, where we have the term "greatness and smallness"[87], paragraphs 8, 9 and 13, the last paragraph determining the meaning of "principle" in paragraph 9, i.e. "Unity". Thomson with regard to the sentence that "the incommensurability of matter is necessary for the coming into existence of these things, the potentiality (or power) of incommensurability is found in them [...]" (in 9), comments that

> Matter is here conceived of Platonically. It is the Indefinite Dyad (cf. *Arist. Metaph.*, 1081a. 14; cf. also 1083b., 34), or The Great and Small (Cf. *Arist. Metaph.*, 987b. 20; cf. also 1085a. 90), which as the material principle of sensibles is, as the *Timaeus* clearly enough says (52a.), space not yet determined by any particular figure and capable of indefinite increase and indefinite diminution.

In the aforementioned paragraph 3, Pappus directly connects Plato's protology with Book X of *Elements* and mathematics.

> If, then, the reason be demanded why a minimum but not a maximum is found in the case of a discrete quantity, whereas in the case of a continuous quantity a maximum but not a minimum is found, you should

85 Cf. *ibidem*, pp. 70–74.

86 As Becker noticed in *Mathematische Existenz*, Eudoxus' theory of proportion from Book V of *Elements* is not constructive. For instance, Definition V. 5 speaks about every possible combination of numbers. Therefore, it is impossible to check the definitional conditions in every case. Eudoxus' theory of proportion is the first non-constructive theory of proportion in Antiquity; cf also Z. Król, *The Development of the Ancient Theories of Proportions*, "Archiwum Historii Filozofii i Myśli Społecznej", 56 (2011), pp. 9–31.

87 W. Thomson explains ([in:] W. Thompson, G. Junge, *op. cit.*, footnote 38) that "the Arabic phrase translated, 'With respect to greatness and smallness', renders the Greek, *κατὰ το`μηῖζον καὶ ἔλαττον* [...]."

> reply that such things as these are distinguished from one-another only by reason of their homogeneity with the finite or the infinite, some of those created things which are contraries of one-another, being finite, whereas the others proceed from infinity. Compare, for example, the contraries, like and unlike, equal and unequal, rest and movement. [Like, equal, and rest, promote (or make for)] finitude [therefore 'finitude' means 'definiteness' — Z.K.]; whereas unlike, unequal, and movement promote (or make for) infinity [i.e. indefiniteness — Z.K.]. And such is the case generally. Unity and plurality, the whole and the parts are similarly constituted. One and the whole clearly belong to the sphere of the finite, whereas the parts and plurality belong to the sphere of the infinite. Consequently one is that which is determined and defined in the case of numbers, since such is the nature of unity, and plurality is infinite (or indefinite). [...] These things, then, are constituted in the manner which we have described[88].

Therefore, the Dyad is not actually an infinite "place" or a kind of infinite arena for geometrical figures, lines *etc*. Instead of an infinite space, ancient mathematicians considered how every mathematical entity is constituted from the highest principles.

However, for almost every modern commentator (there are some exceptions, e.g. H. Weyl, E. Grant), it is doubtless and unquestionable axiom that *Elements* is almost a "complete delineation of the Space which Euclid's geometry is to investigate formally"[89]. Christian Wolf begins his *Anfangsgründe der Geometrie* (1717) with the First Definition: "Geometry is the science of space which is concerned with solid objects in respect of their length, breadth and thickness".

However, the word "space" is absent from *Elements*, in contradiction to the particularity of the ancients to define every basic concept in mathematics, because the foundations of ancient geometry were based on the external principles, higher than mathematical knowledge, of the whole reality[90].

[88] W. Thompson, G. Junge, *op. ci*t.

[89] T.L. Heath, *op. cit*., vol. I, p. 199.

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