ARCHIWUM HISTORII FILOZOFII I MYŚLI SPOŁECZNEJ • ARCHIVE OF THE HISTORY OF PHILOSOPHY AND SOCIAL THOUGHT
VOL. 62/2017 • ISSN 0066–6874

MAREK OTISK

Philosophical Way to God's Wisdom: Arithmetic and the Definitions of a Number in Early Medieval Texts[1]

ABSTRACT: The paper deals with early medieval mathematics (mainly arithmetic) and presents mathematical knowledge as an important tool for human way to God's wisdom. The aim of this paper is focused on the definitions of the subject of arithmetic in early medieval texts created between the late 4th and early 7th century. The aim is to highlight the fact that the traditional definitions of a number (i.e., the subject of arithmetic) correspond with the appropriate topics which exist within arithmetic. If a number is characterised as a discrete quantity, it reflects the classification and typological surveys of the mathematical properties of numbers. If a number is defined as a collection of units, this definition refers to the issue of figural numbers, whereas if the number is marked as the quantity that emerges and then returns to the unit, it is possible to detect the themes of numerical sequences and ratios, including their transfers.
KEYWORDS: Early medieval arithmetic • subject of arithmetic • definitions of number • philosophy of number • Augustine • Boethius

1. Introduction

Between the late 4th and early 7th century numerous Latin texts covered the subject of arithmetic to varying degrees and with different objectives. Meanwhile, statements proposed in those texts became the authoritative basis for the understanding of arithmetical science for a good part of the Middle Ages. The cultivation of the subject matter of arithmetic at (especially early) medieval schools is therefore unthinkable without works by Martianus Felix Capella, Aurelius Augustinus, A.M.T.S. Boethius, F.M.A. Cassiodorus, or Isidore of Seville. All these authors built their works upon the ancient (Neo-Pythagorean and/or Neo-Platonic) foundations of the subject

[1] This paper is a result of the research funded by the The Czech Science Foundation as the project GAČR 17-11657S "Philosophy and Numbers in the Latin Pre-Scholastic Thinking".

matter of arithmetic and considerably aided the continuity of the cultivation of such knowledge in the Latin-Christian intellectual environment.

This paper focuses specifically on these texts regarding the most commonly used definitions of the subject of arithmetic (and, to a limited extent, also on the definition of the subject of mathematics) and attempts to show that these definitions of the subject of this science reflect, quite naturally, the core themes that arithmetic used to deal with. The integral part of this interpretation makes an effort to find ancient sources of the given definitions of the subject of arithmetic (i.e. numbers), thereby supporting the thesis of direct link between ancient Neo-Pythagorean arithmetic, as it was represented, for example, by Theon of Smyrna or Nicomachus of Gerasa, and the early medieval perception of arithmetic.

As the main aim of this paper, based on the analysis of sources, is to demonstrate how the definition of the subject of arithmetic is reflected in the structure of the subject matter of arithmetic, my interpretation will be primarily based on the original texts by the aforementioned authors. These authors considered arithmetic to be a way to true philosophical and divine wisdom and they were simultaneously cultivating ancient philosophical theories concerned with these issues (for instance Ptyhagoreanism, Plato, and Aristotle).

2. *Scientia doctrinalis* and abstract quantity

In the early Middle Ages mathematics as a science was mostly defined as a theoretical or speculative science (*scientia doctrinalis*)[2]. This designation is based on the traditional Aristotelian division of the sciences in which mathematics is ranked among the theoretical or speculative sciences, occupying the middle ground between the first philosophy (the highest science, metaphysics) and physics[3]. Of all the early medieval writers, it is Boethius who, due to his free translation of the treatise *Introduction to Arithmetic* by Nicomachus of Gerasa, was the leading authority on the medieval reception of the subject matter of arithmetic[4]. Boethius, however, does not cover ma-

[2] Cf. for example H.M. Klinkenberg, *Divisio philosophiae*, [in:] *Scientia und ars im Hoch- und Spätmittelalter*, I. Craemer-Ruegenberg & A. Speer. (edd.), Bd. 1. Berlin 1994, pp. 3–19; J.A. Weisheipl, *The Concept of Scientific Knowledge in Greek Philosophy*, [in:] *Melanges a la Memoire de Charles De Koninck*, A. Gagne & T. De Koninck (edd.), Quebec 1968, pp. 487–507; *Idem*, *The Nature, Scope, and Classification of the Sciences*, [in:] *Science in the Middle Ages*, D. C. Lindberg (ed.), Chicago 1977, pp. 461–482.

[3] Cf. for example Aristotelés, *Metaphysica* XI, 7, 1064a–b, I. Bekker (ed.), *Aristotelis Opera omnia*, vol. 2, Berlin 1831 [hereinafter referred to as *Met.*].

[4] Cf. for instance M. Masi, *Boethius' De institutione arithmetica in the Context of Medieval Mathematics*, [in:] *Congresso internazionale di studi Boeziani*, L. Obertello (ed.), Roma

thematical topics and their place among other sciences or disciplines only here, but also in many of his other works. For instance, within a somewhat shorter commentary on Porphyry's *Introduction to Aristotle's Categories* he suggests that the theoretical sciences are called contemplative and speculative according to their subject of interest (i.e., according to the subject of science) and are divided into sciences *de intellectibilibus* (theology), *de intellegibilibus* (mathematics), and *de naturalibus* (concerning the changeable material world)[5]. In the similar context, his first theological treatise (*De trinitate*) identifies three speculative theoretical sciences as follows: *scientia naturalis*, *mathematica*, and *theologia*[6]. This is also consistently expressed by Cassiodorus who regards the mathematics as a theoretical science (*scientia doctrinalis*) since this science, out of all speculative sciences, holds a subject of a theoretical nature in the highest degree (*excellentia*) of concern[7].

Similarly, Cassiodorus also defines the subject of mathematics. Mathematics is, according to him, a science that deals with an abstract quantity (*quantitas abstracta*), that is with a quantity completely irrespective of the bearer of this quantity (abstracted from the material occurrence of quantity), so its subject is exclusively conceived either intellectually or rationally while it ignores further detailed specification of that quantity[8]. Cassiodorus' words are echoed by Isidore of Seville who adds a proposition which can be understood as the further definition of the quantity: he includes

1981, pp. 263–272; P. Kibre, *De Boethian De Institutione Arithmetica and the Quadrivium in the Thirteenth Century University Milieu at Paris*, [in:] *Boethius and the Liberal Arts*, M. Masi (ed.), Bern 1981, pp. 67–80 or J.-Y. Guillaumin, *Boethius's De Institutione Arithmetica and Its Influence on Posterity*, [in:] *A Companion to Boethius in the Middle Ages*, N.H. Kaylor & P.E. Philips (edd.), Leiden 2012, pp. 135–161.

5 A.M.T.S. Boethius, *In Porphyrii Isagogen commentorum editio prima* I, 3, G. Schepss & S. Brandt (edd.), *CSEL* 48, Wien 1906, p. 8,6–8: "Est igitur theoretices, id est contemplatiuae uel speculatiuae, triplex diuersitas atque ipsa pars philosophiae in tres species diuiditur. Est enim una theoretices pars de intellectibilibus, alia de intellegibilibus, alia de naturalibus".

6 A.M.T.S. Boethius, *Quomodo trinitas unus Deus ac non tres dii* 2, [in:] *The Theological Tractates – The Consolation of Philosophy*, H.F. Stewart & E.K. Rand & S.J. Tester (edd. & transl.), Cambridge, Mass. – London 1973, p. 8,5–11: "Nam cum tres sint speculativae partes, *naturalis*, in motu inabstracta *ἀνυπεξαίρετος* […] *mathematica*, sine motu inabstracta […] *theologica*, sine motu abstracta atque separabilis […]".

7 F.M.A. Cassiodorus, *Institutiones* II, praef., 4, R.A.B. Mynors (ed.), Oxford 1961, p. 92,5–8 [hereinafter referred to as *Inst.*]: "Mathematicam vero Latino sermone doctrinalem possumus appellare; quo nomine licet omnia doctrinalia dicere possimus quaecumque docent, haec sibi tamen commune vocabulum propter suam excellentiam proprie vindicavit […]" Cf. *ibidem* II, 3, 21, p. 130,18–19.

8 *Ibidem* II, praef., 4, p. 92, 13–16: "Mathematica vero est scientia quae abstractam considerat quantitatem; abstracta enim quantitas dicitur, quam intellectu a materia separantes vel ab aliis accidentibus, sola ratiocinatione tractamus".

other specifications of quantity, that is, multiple classifications into different categories and types per various criteria (for example, odds and evens, etc.)[9].

Overall, it can be said that early medieval mathematics as such goes beyond the specific definition of quantity because a discrete quantity is not the subject of mathematics but of the individual mathematical branches or sciences. Mathematics itself (a genus superior to individual mathematical sciences) must deal with what encompasses all its subdisciplines without the object of its interest being the same as any of the lower mathematical sciences. In accordance with Aristotle, it is possible to say that all the mathematical sciences have the same subject (i.e. quantity), but each of the mathematical sciences approaches it in a different way[10]. At the same time, it applies that the subject of mathematics is studied rationally regardless of the material bearer of that quantity, in Aristotle's words, it is separated and intangible[11]. Thus, as it is a subject separated from the material, as an abstract quantity, it appertains to persistence and stability[12].

Aristotle's assertion is taken up by Boethius who, in *Introduction to Arithmetic*, mentions that the subject of mathematics is a quantity that is not subject to change and is, therefore, stable (*immutabile*)[13]. Without knowledge of this subject we can never attain wisdom, as every philosopher must begin the search for knowledge from the mathematical sciences[14]. Under the mathematical sciences, all *quadrivium* disciplines, as Boethius called them[15], are included arithmetic, music, geometry (including stereometry), and astronomy – this tradition can be traced back at least as far as the Pythagorean school[16].

[9] Isidori Hispalensis Episcopi *Etymologiarum siue Originum libri XX* III, praef., l. 1–5, W.M. Lindsay (ed.), Oxford 1911 [hereinafter referred to as *Etym.*]: "Mathematica Latine dicitur doctrinalis scientia, quae abstractam considerat quantitatem. Abstracta enim quantitas est, quam intellectu a materia separantes uel ab aliis accidentibus, ut est par, inpar, uel ab aliis huiuscemodi in sola ratiocinatione tractamus".

[10] Aristotelés, *Met.* VI, 1, 1026a.

[11] Cf. for example Aristotelés, *De anima* III, 7, 431b, I. Bekker (ed.), *Aristotelis Opera omnia*, vol. 1, *op. cit.* or *idem*, *Physica* II, 2, 193b–194a, I. Bekker (ed.), *Aristotelis Opera omnia*, vol. 1, *op. cit.*

[12] *Idem*, *Met.* VI, 1, 1026a.

[13] A.M.T.S. Boethius, *De institutione arithmetica* I, 1, H. Oosthout & J. Schilling (edd.), *CCSL* 94A, Turnhout 1999, p. 9,8–19 [hereinafter referred to as *De inst. arith.*].

[14] *Ibidem*, I, 1, p. 11,47–50: "Quod haec qui spernit, id est has semitas sapientiae, ei denuntio non recte philosophandum, siquidem philosophia est amor sapientiae, quam in his spernendis ante contempserit".

[15] *Ibidem*, I, 1, p. 9,6–7 or *ibidem*, p. 11,64.

[16] See e.g. Archytas of Tarentum (for example fragment 47 B 1 according to *Die Fragmente der Vorsokratiker*, H. Diels & W. Kranz (edd.), Berlin 1952) or Plato (for instance Platón, *Respublica* VI, 6–12, 521c–531c, J. Burnet (ed.), *Platonis Opera*, vol. 4, Oxford 1907

Boethius's *Introduction to Arithmetic* in the opening chapter of the first book includes a detailed analysis of an interaction between the various subjects of the mathematical sciences on the basis that all the sciences somehow deal with a quantity. Aristotle already distinguished between discontinuous or disjunctive quantity (e.g., a number) and a continuous (e.g., a line, or a solid)[17], that is, respectively, the quantity in which the individual parts themselves are not related (e.g. a number) and the quantity in which the parts are directly linked together (e.g., a line, or a solid). The former is referred to as a plurality, while the latter is reffered to as a magnitude[18]. Boethius uses the following terms for these distinctions – disjunctive (*disiuncta*) quantity, that is multitude (*multitudo*), and continuous (*continua*) quantity, that is magnitude (*magnitudo*)[19]. Also, in accordance with Aristotle, Boethius distinguishes between a quantity which occurs by itself (*per se*), for example, a number, geometrical shape, etc., and a quantitative determination, which is related to something else (*ad aliud*), for example, numerical ratios, musical intervals, etc. It is also possible to consider quantity as something absolutely unchangeable and constant, for example, a geometric shape, numerical value, etc., or as something that is in constant and perfect circular motion, for example, the orbits of celestial bodies[20].

For Boethius, these are the distinctions from which the proper division of mathematics grows. While mathematics as such explores an undefined, abstract quantity, arithmetic examines the number itself (*multitudo per se*), music studies multitude in relation to another (*multitudo ad aliud*), geometry investigates stable magnitude (*magnitudo stabilis*) and astronomy deals with movable magnitude (*magnitudo mobilis*). This also indicates a hierarchical arrangement in the mathematical sciences – the most perfect and most basic is arithmetic, because its subject (i.e., the number) is necessary for all other mathematical sciences; geometry is listed in second place and, although it needs numbers for its art, it provides the necessary basis for astronomy; third

[hereinafter referred to as *Resp.*]) and Aristotle (Aristotelés, *Analytica posteriora* I, 13, 79a, I. Bekker (ed.), *Aristotelis Opera omnia*, vol. 1, *op. cit.*). Cf. also e.g. Martianus Capella, *De nuptiis Philologiae et Mercurii* VI–IX, 567–1000, J. Willis (ed.), Leipzig 1983, p. 201–368 [hereinafter referred to as *De nupt.*]; Cassiodorus, *Inst.* II, 4–7, p. 132–162; Isidorus, *Etym.* III, 1–71 or Aurelius Augustinus, *De ordine* II, 5, l. 6–8, W.M. Green & K.D. Daur (edd.), [in:] *idem*, *Contra academicos; De beata vita; De ordine; De magistro; De libero arbitrio*, *CCSL* 29, Turnhout 1970 [hereinafter referred to as *De ord.*].

17 Aristotelés, *Categoriae* 6, 4b, I. Bekker (ed.), *Aristotelis Opera omnia*, vol. 1, *op. cit.* [hereinafter referred to as *Cat.*].

18 *Idem*, *Met.* V, 13, 1020a.

19 Boethius, *De inst. arith.* I, 1, p. 10,23–30.

20 *Ibidem*, p. 10,31–38.

position is occupied by music, which is dependent only on arithmetic; and finally there is astronomy, whose art is dependent not only on arithmetic, but also on geometry[21].

3. Number as the subject of arithmetic

Arithmetic is thus presented by Boethius as the highest of the mathematical sciences. With explicit references to Plato[22], he points out that arithmetic is the mother of all sciences, since without numbers no science could function, including philosophy[23]. Numbers represent the highest genus, whose existence is necessary for the existence of all lower genera. If any lower species became extinct, it would have no effect upon the existence of a superior kind, but if this superior kind died out, then the inferior species would necessarily have to disappear. Thus, if numbers form the highest genus, then everything else is dependent on their existence: If numbers perished, everything subordinated to them would cease to exist, for example, the Earth, mankind, etc.[24].

Holy Scripture also provides a very good opportunity to elucidate the key role of numbers which substantiate the Divine basis of the created world, as the Book of Wisdom states that God created everything in compliance with the measure, number, and weight (*mensura*, *numerus*, *pondus*)[25]. Thus, for Boethius numbers become the very thoughts of God. They are the ideas and forms in the mind of God through which everything is created[26]. Numbers form the basis of the order built by God, according to which everything is arranged in the universe. Thus, numerical ratios are those maintaining harmony within the created world and, at the same time, refer to a harmonious relationship between the Creator and the Creation[27].

[21] *Ibidem*, p. 12,73–14,130.

[22] Platón, *Resp.* VII, 10, 527d–e.

[23] Boethius, *De inst. arith.* I, 1, p. 11,67–76. Cf. e.g. Platón, *Resp.* VII, 6, 522c or *ibidem*, VII, 8, 525a–526a.

[24] Boethius, *De inst. arith.* I, 1, p. 12,78–90, resp. *ibidem* I, 2, p. 14,2–15,23.

[25] *Sap.* 11,20. Cf. for example Aurelius Augustinus, *De libero arbitrio* II, 8, 20–12, 34, W.M. Green & K.D. Daur (edd.), [in:] *idem*, *Contra academicos; De beata vita; De ordine; De magistro; De libero arbitrio*, *op. cit.* or *idem*, *De Genesi ad litteram libri duodecim* II, 1, [in:] *idem*, *De Genesi ad litteram libri duodecim*; *De Genesi ad litteram imperfectus liber, Locutionum in Heptateuchum libri septem*, J. Zycha (ed.), *CSEL* 28/1, Wien 1894 etc.

[26] Boethius, *De inst. arith.* I, 2, p. 12,75–79: "Haec enim cunctis prior est, non modo quod hanc ille huius mundanae molis conditor deus primam suae habuit ratiocinationis exemplar et ad hanc cuncta constituit, quaecumque fabricante ratione per numeros assignati ordinis inuenere concordiam […]". Cf. *ibidem*, I, 2, p. 14,3–4.

[27] *Ibidem* II, 1, p. 93,2–94,16.

Therefore, as the arithmetic is always primarily concerned with numbers, it can properly be called the science of numbers. Cassiodorus in *Institutiones* explains the proper name of arithmetic by the fact that this science deals with numbers[28]. Isidore of Seville echoes this when he writes that arithmetic is the doctrine of numbers, because the Greeks call the number ἀριθμός[29].

However, as Plato already mentioned (for example, in the dialogue *Gorgias*)[30], within arithmetical traditions it is possible to work with numbers in different ways, namely by distinguishing arithmetic itself as a doctrine of the properties of numbers (i.e. ἀριθμητικὴ) and arithmetic as an instruction to arithmetical operations with concrete numbers, i.e. computation (λογιστικὴ). In *The Republic* Plato states that the skill of computation is of secondary importance and not worthy of a philosopher's interest (especially regarding mercantile calculations). Nevertheless, such knowledge also plays its irreplaceable part in physical training. Arithmetic is more important and higher science through which one can comprehend the fact that the numbers themselves or their source (unit) can direct the human soul to the immutable and permanent spheres, but the arithmetical or mathematical properties of numbers are also essential, that is, theoretical arithmetic itself[31]. In addition to practical arithmetic (computation) is thus necessary for theoretical arithmetic to distinguish symbolic arithmetic (a particular form of numerology: the interpretation of the symbolic meaning of numbers) and theoretical arithmetic itself (that is, the interpretation of the mathematical properties of numbers). Perhaps the clearest distinction between the two traditions of theoretical arithmetic is evident in the work of Martianus Capella in which arithmetical treatises on the mathematical properties of numbers are preceded by a brief explanation of the symbolic meaning of the numbers one to ten[32].

All these assessments and resolutions imply that the subject of arithmetic is number as quantitative determination. From antiquity, the

[28] Cassiodorus, *Inst.* II, 4, 2, p. 133, 11: "Arithmetica vero dicitur eo quod numeris praeest".

[29] Isidorus, *Etym.* III, 1, 1, l. 14–16: "Arithmetica est disciplina numerorum. Graeci enim numerum *ἀριθμόν* dicunt". The translation of the Greek term for number into the Latin by a term *numerus* was common in the early Middle Ages, and already Aurelius Augustinus frequently operated with it in his works – see e.g. Augustinus, *De ord.* II, 14, 40 or *idem*, *De musica libri VI* III 1, 2, J.-P. Migne (ed.), *PL* 32, Paris 1841, c. 1115.

[30] Platón, *Gorgias* 451 b–c, J. Burnet (ed.), *Platonis Opera*, vol. 3, Oxford 1906.

[31] *Idem*, *Resp.* VII, 8, 525a–526c; cf. e.g. M. Masi, *Arithmetic*, [in:] *The Seven Liberal Arts in the Middle Ages*, D.L. Wagner (ed.), Bloomington 1983, pp. 147–167.

[32] Martianus Capella, *De nupt.* VII, 731–742, pp. 262–269; respectively *ibidem* VII, 743–801, pp. 269–301.

idea has existed that every quantity is expressed by a unit or number[33]. Each number is derived from the unit, so the unit can be described as the cause of the number, the mother of all numbers, their essential source[34]. Thus, the unit cannot be regarded as a number, and the relationship between numbers and units is similar to the relationship between the Creator and the Creation. Boethius says that One is the foundation of God and the absolute Good, from which immediately arise dualities (the number two) – for example, darkness and light, Heaven and Earth, etc. If everything is related to its root cause (i.e., until everything is contained in a number or determined by a numerical ratio), then it is also participating on the supreme Good. The moment anything deviates from numbers, it also turns away from good, units, and thus from God himself[35].

3.1 Number as a collection of units and figural numbers

Nicomachus of Gerasa in *Introduction to Arithmetic* mentions the three most commonly used definitions of the number: "Number is limited multitude or a combination of units or a flow of quantity made up of units [...]"[36]. Numbers can be described as a discrete (limited) quantity, or as a combination of units, or as an infinite set that originates from the unit (and apparently returns back to it). Though these definitions may seem at first glance quite different, we can find a unifying line in them which also corresponds with the late ancient and early medieval interpretations of arithmetical subject matter, that is, with contents of treatises on the mathematical properties of numbers.

The most widely used definition of the number in the early Middle Ages was Nicomachus' second assessment, which states that the number is a combination or arrangement. For example, Boethius in his loose translation of the Nicomachean treatise, among others, indicates that numbers are a collection of units: "A number is a collection of unities [...]"[37].

Additionally, other early medieval texts speak of the number as arrangement or orderings (*congregatio, compositio, constitutio*) of units (*mona-*

[33] For example, Aristotelés, *Met.* X, 1, 1052b.

[34] Cf. eg. Boethius, *De inst. artih.* I, 17, p. 45,106.

[35] *Ibidem* I, 32, p. 80, 3–19.

[36] Nicomachi Geraseni Pythagorei *Introductionis Arithmeticae libri II* I, 7, 1, R. Hoche (ed.), Leipzig 1866 [hereinafter referred to as *Intr. arith.*], p. 13,7–8 (English translation: Nicomachus of Gerasa, *Introduction to Arithmetic*, M.L. D'Ooge (transl.), New York – London 1926, p. 190).

[37] Boethius, *De inst. arith.* I, 3, p. 15,2–3: "Numerus est unitatum collectio [...]" (English translation: M. Masi, *Boethian Number Theory. A Translation of the De Institutione Arithmetica*, Amsterdam 1983, p. 76).

des, unitates)[38]. By this definition a specific number can be imagined as a group of units. The value of the number is given by the amout of units. The number five is, therefore, represented by five mutually arranged individual points or five units whose sum is five.

This definition was already very popular in antiquity. According to Iamblichus, we can find the origins of such a designation of the number already in Thales of Miletus[39], although Aristotle's report that numbers were defined in such a way by Pythagoreans can be regarded as more probable[40]. The ancient popularity of perceiving numbers this way is evidenced by the presence of the same definition in Euclid's *Elements*[41].

It seems that the whole tradition of defining numbers as a collection of units is very close to the traditional arithmetical practice of the visual representation of numerical values using points that can be organized following different rules. This arithmetical topic is generally described as figural numbers – divided in accordance with the amount of directions in which these points (whose sum constitutes the visual appearance of numbers) are moved in or are added to them. This allows us to distinguish (I.) linear numbers (points moving in one direction – i.e., along the line, where forward and backward movement can be differentiated) – in other words, line segments; (II.) plane numbers (points moving along two lines – backward and forward, respectively, up and down) – in other words, moving line segments, that is, surface formations (triangles, rectangles, etc.); and (III.) solid number (points that can go in three directions – besides moving forward and backward, respectively up and down, there are added movements to the right and left) – in other words, solid figures (pyramids, cubes, polygons, etc.)[42]. It seems that the definition of numbers as a collection of units almost suggests the idea of the figural character of numbers (the topic linked already with Pythagoreans) which is essentially dictated by the amount of points and the way they are arranged in a given number.

[38] Cf. Martianus Capella, *De nupt.* VII, 743, p. 269,15 (*congregatio monadum*); Cassiodorus, *Inst.* II, 4, 2, p. 133,12 (*ex monadibus multitudo composita*) or Isidorus, *Etym.* III, 3, 1, l. 1–2 (*multitudo ex unitatibus constituta*).

[39] Iamblichi *In Nicomachi arithmeticam introductionem*, H. Pistelli (ed.), Leipzig 1894 [hereinafter referred to as *In Intr. arith.*], p. 10,8–9.

[40] Aristotelés, *Met.* I, 5, 986a.

[41] Euclidis *Elementa* VII, def. 2, I.L. Heiberg & H. Menge (edd.), vol. 2, Leipzig 1884 [hereinafter referred to as *Elem.*], p. 184,4–5.

[42] Cf. e.g. Nicomachus, *Intr. arith.* II, 6–17, p. 82–112; Boethius, *De inst. arith.* II, 5–30, p. 110–152 or Isidorus, *Etym.* III, 7, 1–6. Euclid deals with this issue only marginally, and differently to certain extent – cf. e.g. definition of plane (and then square) numbers or solid (and then cube) numbers – Eukleidés, *Elem.* VII, def. 16–20, pp. 186,14–188,4.

3.2 Number defined as a quantity and the classification of numbers

It is probably not a coincidence that the first definition by Nicomachus represented the number as a quantity specified in a certain way. Similarly, early medieval texts on arithmetic do not start explaining the subject matter of arithmetic by figural numbers but by using several classifications and typologies of numbers. This subject matter, by its nature, apparently corresponds precisely with the definition of numbers which Cassiodorus gives in *Institutiones* in which he mentions that numbers may be understood as a discrete quantity (*quantitas discreta*)[43].

In this respect, numbers are a specific quantity. Each number is explicitly shown in a particular definition, for example, the number five is a quantity which is limited to the value of five. Each number is therefore fixed and has nothing in common with another number. The source of Cassiodor's definition of the number was probably Boethius' *Introduction to Arithmetic*, in which he states that numbers are always discrete (*discreta*) and may enter into mutual relationships as such (ratios), creating the orderliness and harmony of the world[44]. In the same way, several ancient authors also expressed the possibility of defining a number: for example, Aristotle in *Categories* or *Metaphysics*[45], or (according to later Iamblichus' references) Eudoxus of Cnidus[46].

The aforementioned explanation clearly reflects the traditional view of understanding the definition as such[47]. If mathematics is a genus superior to individual mathematical arts (including arithmetic), and if the subject of mathematics is abstract quantity, then the subordinate kind of mathematics, meaning here arithmetic, also consistently takes quantity for its subject – discrete as opposed to undiscrete (abstract). In this case the quantity plays the role of a superior genus and the definition of such a quantity represents a specific difference.

Quantities may be determined in various ways. It has already been suggested that the given definition may determine the actual value of a specific number. But quantities can also be determined by other criteria which enable the creation of all sorts of classification account of numbers. In particular, the typology of numbers represents the subject matter of

43 Cassiodorus, *Inst.* II, 4, 2, p. 133,8.
44 Boethius, *De inst. arith.* I, 2, p. 15,12–15.
45 Aristotelés, *Cat.* 6, 4b or *idem*, *Met.* V, 13, 1020a.
46 Iamblichos, *In Intr. arith.*, p. 10, 17–18.
47 Cf. for example Aristotelés, *Topica* I, 5, 101b, I. Bekker (ed.), *Aristotelis Opera omnia*, vol. 1, *op. cit.*

arithmetic, preliminarily solved in the late antiquity. The most frequent methods of classifying numbers, dealt with in fact in all the early medieval texts on arithmetic, are the classifications of numbers into odd and even, respectively their subtypes (even times even, even times odd, odd times even, if necessary also odd times odd, respectively into primes, and composite or intermediate numbers), or into abundant, diminished and perfect numbers[48].

The numerical properties determined by the given identifications of quantity, that is, typological traditions of late antique arithmetic, have been used widely since the early Middle Ages to explain the significance of certain numerical values that play an important role, for example, in the Bible[49]. Above all, we should mention Saint Augustine, who, for example, in *De civitate Dei*, among other things, explains why the act of creation took six days[50]. The number six is a perfect number, as it is given by its fractions, that is, six can be completely divided by the numbers three, two and one (i.e., one half, one third and one sixth arise) and the sum of these divisors is equal to the number six. Augustine adds that similar numbers are very few, as most numbers are diminished (the sum of denominators, respectively divisors, is smaller than the value of divided numbers – e.g., the number nine or ten) or abundant (the sum of denominators, respectively divisors, is greater than the value of the divided numbers – e.g. the number twelve)[51]. As Nicomachus states, with diminished numbers we experience the failure of their parts to create the original unit, while in the case of abundant numbers a situation occures that their parts create more than the sum of the original unit occurs[52]. Only in the case of perfect numbers their parts add up to a whole, and these numbers are very rare (in the early Middle Ages only the first four perfect numbers were used 6, 28, 496 and 8128, although the algorithm for finding others was known)[53]. For Augustine, the arithmetical perfection of the number six is what determined the number of days in which God realized the ultimate act of creation.

[48] Cf. e.g. Nicomachus, *Intr. arith.* I, 7–16, p. 13–44; Boethius, *De inst. arith.* I, 3–20, p.15–54. See also (in some cases slightly differently) Eukleidés, *Elem.* VII, def. 6–10, pp. 184,11–186,2 or *ibidem* VII, def. 23, p. 188,11–12.

[49] For more details see e.g. H. Meyer & R. Suntrup (edd.), *Lexikon der mittelalterlichen Zahlenbedeutungen*, München 1987.

[50] Aurelius Augustinus, *De civitate Dei libri XXII* XI, 30, B. Dombart & A. Kalb. (edd.), vol. 2, *CCSL* 48, Turnhout 1955 [hereinafter referred to as *De civ. Dei*], l. 1–11.

[51] *Ibidem* XI, 30, l. 12–30.

[52] Nicomachus, *Intr. arith.* I, 14–15, p. 36–39.

[53] Boethius, *De inst. arith.* I, 20, p. 51,7–9: "Sunt autem perfecti numeri intra denarium numerum VI, intra centenarium XXVIII, intra millenarium CCCCXCVI, intra decem milia $\bar{V}\bar{I}\bar{I}$CXXVIII".

In the very next chapter of *De civitate Dei* Augustine moves to the number seven, which is marked as the number of completion since it corresponds to the six days of creation and the seventh day of rest. Seven can therefore be properly considered a reference to everything finished and created, that is, everything that has its origin and its harmony in numbers or numerical ratios. Therefore, this figure can be conceived as the symbol of all numbers. The reasons for designation of seven as an expression of all the numbers is not only given by biblical allusions, but they are again justified primarily by mathematical properties of numbers. Seven is in fact the sum of two numbers, which together include all numbers – odd numbers are represented by the lowest odd number (i.e., three), and even numbers by the lowest even (and at the same time even times even) number (i.e., four)[54]. The preeminence of the number three between odd numbers is probably universally understood (the number one is not a number, the number two is not odd), but in case of the number four doubts may arise, as it may mean that the lowest even number is the number two.

However, this statement concerning the number two in late antiquity and the early Middle Ages was linked to several problems. Odd numbers were usually defined as incompetent of division into two equal integers, because one component will always be one unit larger or smaller. Even numbers are then defined as those that can be divided into two equal integers[55]. Neither of these definitions is valid for the number two – unlike odd numbers, it may be divided into two identical components, but not into the same two numbers, as required by the definition of even numbers (a unit is not a number, but a source of numbers). Should anyone still maintain that the number two is an even number because it can be divided into two equal halves, another problem soon arises. In the typology of even numbers, the number two would match the definition of an even times even numbers (i.e., such numbers that can always be divided into equal halves until we reach a unit, namely the numbers 4, 8, 16, 32, etc.) and the definition of even times odd numbers (i.e., such even numbers can be divided into two equal halves, although this division does not give birth to an even number, so it can no longer be divided into equal halves – these are, for example, the numbers 6, 10, 14, 18, etc.)[56]. For these reasons the number two is often considered

[54] Augustinus, *De civ. Dei* XI, 31, l. 1–15.

[55] Cf. e.g. Boethius, *De inst. arith.* I, 3, p. 16,5; Nicomachus, *Intr. arith.* I, 7, p. 13, 10–11 or Isidorus, *Etym.* III, 5, 2, l. 20–21 etc. See also different definition by Euclid – Eukleidés, *Elem.* VII, def. 8, p. 184,14–15.

[56] Cf. e.g. Nicomachus, *Intr. arith.* I, 8–9, p. 14–21; Boethius, *De inst. arith.* I, 9–10, p. 21–30 or Isidore, *Etym.* III, 5, 3–4, l. 23–4. See also different definition by Euclid – Eukleidés, *Elem.* VII, def. 9, p. 184,16–17.

a somewhat strange and controversial figure that cannot be safely in the usual categories of discrete quantity, so the first really definite even number becomes the number four.

3.3 Number as a stream coming out of a unit (and returning back to it again) and numerical sequences

The third definition of numbers was most comprehensively developed by Martianus Capella[57] who, in addition to widespread definition of numbers as a collection of units, also states that numbers can be understood as a certain multitude which has its source in the unit to which it returns again: "A number is […] a multitude proceeding from a monad and returning to a monad"[58].

He also included this way of characterizing numbers in his description of Lady Arithmetic who appears in front of the gods during a wedding congregation and whose appearance frightens the celestials. One of the main causes of this horror is the mysterious, barely visible beam that emanates from venerable Arithmetic's forehead. It then spreads out and expands before shrinking again, and eventually returns to its source[59]. This mysterious beam is the image of numbers that have their origins in units. All the multitude of numbers is dependent on this source, whereas individual numbers are connected to each other (mainstream beam), but can also enter various relationships (branching beam) where there are fixed relations between the numbers dependent on previous figures and their relations. Numbers, on the one hand, arise from a single origin and may proceed to infinity, but they can return to their primary source and mother of all the numbers (the shrinking of the beam).

This definition of numbers also appears in Boethius[60] in a partially modified form but even in this case it can be traced back to its ancient origins. John Stobaios recorded passages from the works of a Neo-Pythagorean thinker Moderatus of Gades, which contain characteristics indicating that numbers are what emerges from the unit, and will return again to it[61].

[57] For more details about Martianus Capella see for instance W.H. Stahl, *The quadrivium of Martianus Capella. Latin traditions in the mathematical sciences, 50 B.C.–A.D. 1250*, (Martianus Capella and the seven liberal arts, vol. 1), New York 1971.

[58] Martianus Capella, *De nupt.* VII, 743, p. 269, 15–16: "Numerus est […] a monade veniens multitudo atque in monadem desinens". (English translation: Martianus Capella, *The Marriage of Philology and Mercury*, (Martianus Capella and the seven liberal arts, vol. 2), W.H. Stahl & R. Johns & E.L.Burge (transl.), New York 1977, p. 285).

[59] *Ibidem* VII, 728–729, pp. 260–262.

[60] Boethius, *De inst. arith.* I, 3, pp. 15, 2–16,1: "Numerus est […] quantitatis aceruus ex unitatibus profusus."

[61] Ioannis Stobaei, *Eclogarum physicarum et ethicarum libri duo* I, 1, 8, A. Meineke (ed.), vol. 1, Leipzig 1860, p. 5,13–15.

The definition of numbers draws attention to the infinite series of numbers and to the interdependence and dependence of numbers on each other. The given definition indicates that numbers have not only properties of their own, but they receive certain characteristics thanks to their relations to other numbers. It can be assumed that this definition refers, among other things, to the next big topic of theoretical arithmetic: the properties of numbers, insofar as they are related to other numbers, that is, the relative properties of numbers (mainly numerical ratios). Ancient and medieval arithmetic in this regard distinguishes between the numbers which have the same value (e.g., a dozen and a dozen, an ell and an ell, etc.) and those which do not possess the same value (e.g., a dozen and threescore, a foot and an ell, etc.). Unequal (*inaequalis*) numbers are then divided into those in which a larger number is compared with a lesser (ratios derived from multiples: that is, multiples, superparticular numbers, superpartient numbers, superparticular multiples and superpartient multiples), and numbers for which a smaller number is compared to a larger (ratios derived from the divisors: that is, divisors, subsuperparticular numbers, subsuperpartient numbers, subsuperparticular divisors and subsuperpartient divisors)[62].

All these ratios arise from the equality (*aequalitas*), which is provided by unity (1 : 1 ratio). It creates the order and rules that are present in this world, enables, in compliance with the fixed mathematical rules, the reconstruction of the creation of all ratios, and when reversed, it also points the way back to unity and equality – the goal of all created things. Thus, numerical sequences determined by the specific ratio appear in the forefront of arithmetical interest, which is traditionally the climax of the arithmetical learning – particularly arithmetic, geometric and harmonic proportion[63].

4. Conclusion

It seems that early medieval understanding of the subject matter of arithmetic is unambigously linked to the Neo-Pythagorean (Nicomachean) tradition of the cultivation of this science. While the subject of mathematics is an undetermined abstract quantity, various mathematical sciences have quantity defined by their subject in a certain way. To achieve this, Boethius

[62] Cf. e.g. Nicomachus, *Intr. arith.* I, 17–II, 5, pp. 44–82; Boethius, *De inst. artih.* I, 21–II, 3, pp. 54–105 or Isidorus, *Etym.*, III, 6, 1–13. See also Eukleidés, *Elem.* VII, def. 3–4, p. 184,6–8.

[63] Cf. Nicomachus, *Intr. arith.* II, 21–27, pp. 119–140 or Boethius, *De inst. arith.* II, 40–50, pp. 172–213. See also different definition of proportion in Eukleidés, *Elem.* VII, def. 21, p. 188, 5–7.

deployed the distinctions such as *multitudo* and *magnitudo*, *per se* and *ad aliud*, respectively *stabilis* and *mobilis*; thereby he defined and hierarchically organized four basic mathematical sciences: arithmetic, geometry, music and astronomy.

The very subject of the first mathematical science, that is, arithmetic, is thus marked by the concretization (definition) of quantity into numbers. However, in the early Middle Ages more definitions of numbers were used, not necessarily determined only by different approaches to the specification of the nature and essence of numbers. The reason for the different definitions could be the topic itself, which was discussed in the context of the early medieval compendia on arithmetic, and texts inquiring into arithmetic.

If numbers are characterized as a discrete quantity (*quantitas discreta*), then it highlights the direct link to the basic mathematical properties of numbers and their typology, that is, the first major topic that the former arithmetic dealt with. If numbers are defined as the sum of the units (*collectio unitatum*), it suggests the idea of figurative numbers – the second broad topic of early medieval textbooks on arithmetical knowledge. If numbers are understood as meaning a stream that springs from a beginning (*a monade veniens*), it can gradually expand to infinity, but, at the same time, it branches out and eventually returns back to its source (*in monade desinens*), then it fully corresponds with the issue of numbers, how they are related to other numbers, that is, to the relative properties of numbers and numerical ratios that establish numerical sequences etc., that is the next and, in fact the last, major topic of theoretical arithmetic, how it was cultivated in (early) medieval schools.

MAREK OTISK – profesor w Zakładzie Filozofii Uniwersytetu Ostrawskiego oraz pracownik naukowy w Zakładzie Studiów nad Myślą Starożytną i Średniowieczną Czeskiej Akademii Nauk w Pradze. Jego zainteresowania naukowe oscylują wokół początków filozofii scholastycznej między X i XII stuleciem, szczególnie zaś dotyczą *quadrivium* oraz tematów metodologiczno-logicznych, gramatyczno-dialektycznych i metafizycznych.

MAREK OTISK – an Associate Professor at the Department of Philosophy, Faculty of Arts, University of Ostrava, and a research worker at the Department for the Study of Ancient and Medieval Thought, Institute of Philosophy, the Academy of Sciences of the Czech Republic, Prague. His research concerns the beginnings of scholastic philosophy between 10th and 12th century, especially focusing on the *quadrivium* and on methodological-logical, grammatical-dialectical and metaphysical topics.